Scattered Information on a Speculative Market

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Abstract

A very simple static, rational expectations, closed-form model is built as an alternative to Grossman-Stiglitz [1980]. It is the general framework for a systematic investigation of the way randomly distributed individual characteristics swamps informations relevant to the stock market equilibrium.The most important of those "noises" are the individual endowments in the numeraire good. Together with the endogenous quality of private signals, they allow a better understanding of the volatility of stock prices and of the Grossman-Stiglitz paradox. The real issue is not the "informational efficiency" of the market, but the direct computation of the rational expectations equilibrium by putting in common anything anybody knows or observes in the economy.

"Explaining a model of efficient capital markets by writing for the thousandth time 'P_t given I_t, where I_t is all the information' does not advance understanding. If it didn't much help to make Eugene Fama's work clear when he first uttered it, why suppose it will enlighten someone now?"

D.N. McCLOSKEY [1987] p. 24

The notion of "information revelation by the equilibrium price" still needs to be made more precise. Actually, a powerful critique of rational expectations equilibrium, as a tool for investigating information acquisition behaviours, is that, "in order to understand the value of and incentives to acquire asymmetric information, one has to model the mechanism through which prices are formed" (M.O. Jackson, J. Peck [1999] p. 603). However, recent papers on continuous double auction mechanisms (D. Friedman [1984]) have shown that by a "miracle" (V.L. Smith [1982]) the walrasian equilibrium is eventually set up, even with "zero-intelligence traders" (D.K. Gode, S. Sunder [1993]). Finally there is a case for studying the existence and properties of a walrasian equilibrium without paying much attention to the joint complete analysis of a realistic pricing mechanism (tatonnement, quasi-tatonnement, market games, dealers markets, etc.). Since the choice of such a mechanism is immaterial, I pick up the more straightforward one (a clearing house), leading easily to the introduction of others price revealing economic variables, without standing to far from market practices.

Concerning modelisation techniques, ever since the seminal 1980 article, the Grossman-Stiglitz approach (*i.e.* negative exponential utilities with gaussian random variables) has been the only available alternative in the field of asymmetric information (cf. for instance A. Kyle [1985] or C. Hsu [1998]). However using other HARA elementary utility functions, with income effects (as in S.F. LeRoy , C.J. LaCivita [1981]), allows the introduction of important "noises" preventing the revelation from being "full" and, again, points to other price revealing economic variables.

Along those lines, a handful of simple models are presented to illustrate an approach, alternative to Grossman-Stiglitz [1980] and to Ausubel [1990], leading to a better understanding of the reasons for *partial* revelation and of the *direct* computation of the rationally expected, partially revealing, equilibrium price.

1 The model

We are considering the simplest walrasian exchange economy with only one good and one financial asset ("stock"). It is a *static* one*:* each individual takes a single decision and all the decisions are simultaneous. The temporal framework is the most parcimonious: everything happens between *a priori* and *a posteriori*, two infinitely close time points, but lots of things happen successively in the twinkling of an eye:

- each individual observes the value taken by his *private* signal;

- he figures out an individual demand function for the financial asset;

- all the individual demand functions are gathered and centrally processed by a "clearing house" in order to balance the market for the financial asset;

- all transactions are settled at the posted equilibrium price;

- a *state of nature* comes true;

- it is the condition for the sharing of a profit between the shareholders;

- each individual consume all the good at his disposal (either kept from the start, or gained from selling assets or earning dividends).

Three remarks are not out of order:

- some of the good is alloted as initial endowments to the individuals right from the start, the rest happens to be created *a posteriori* only to be shared as dividends;

- there is no true time horizon, no period from date zero to date one, no interest rate and, of course, no link ("production function") between inputs and outputs;

- the seven-item summary of what happens between *a priori* and *a posteriori* does not include "observing the equilibrium value of the asset price and using the information that it reveals". Of course that is so because the clearing house (thereafter CH) provides a shortcut to that episode. When he supplies his "buy order" to the CH, the individual, who has just observed the value taken by his private signal, is rationally expecting the possible values of the price about to be posted by the CH. Actually his (net) demand function is a list of quantities (more precisely of proportions), each of them being associated with one of those possible equilibrium values, that conditions it. The information revelation is virtual and beforehand the individual makes himself ready to cope with every alternative.

We will proceed very simply by considering an economy in which

- one good is consumed only *a posteriori,* we call it the numeraire*;*

- one of H possible states of nature comes true *a posteriori;*

- the only financial asset is the outstanding stock of an enterprise sharing its *a posteriori* profit between its stockholders. This uncertain profit Q_h is conditioned by the state h ($h \in \{1, \ldots, H\}$);

- N individuals are indexed by i ($i \in \{1, \ldots, N\}$).

Individual i maximize a Von Neumann utility with elementary utility $u_i\left(W_i^+\right)$, where W_i^+ is the total amount of good ultimately available to him: he consumes his whole *a posteriori* wealth.

A priori, individual i is endowed with an amount ω^i of numeraire good and a proportion θ_i^- of the enterprise stocks; *a posteriori*, once the transactions have been settled on the stock market, the proportion held becomes θ_i^+. Let v be the equilibrium market value of the entreprise (the value of the whole of its stocks), the *a posteriori* wealth of i in the state h will be:

$$W_i(h) = \omega^i + \theta_i^- v + \theta_i^+ (Q_h - v). \tag{1}$$

The "decision" taken by individual i in order to maximize his expected utility is the choice of the net demand function, written as a proportion

$$\theta_i = \theta_i^+ - \theta_i^- = \delta_i(v), \tag{2}$$

to be forwarded to the CH. Just before taking his decision, the individal is "personally" (and imperfectly) informed but he does not know exactly either the exact state h about to happen, or the value of v about to clear the market. Before this value is set and published by the CH, the walrasian equilibrium price, as seen by individuals, is a random variable and we will suppose it to be rationally expected. Before going further and being more specific, there is no harm in supposing this random variable to take a *finite* number of possible values v_m (with $m \in \{1, \dots, M\}$). Therefore the individual's aim is to maximize the expected utility

$$E_i(u_i) = \sum_{m,h} \pi_i(v_m, Q_h) u_i \left[\omega^i + \theta_i^- v_m + \theta_i^+ (Q_h - v_m)\right], \tag{3}$$

where $\pi_i(v_m, Q_h)$ is the probability ('belief') assigned by i to the couple of values (v_m, Q_h). As we will see, this belief is conditional on what i privately observes just before taking his decision.

2 Information and equilibrium

In the kind of simple one-market economy we are analyzing, to look for the (unique) equilibrium is to compute the price level balancing supply and demand. When there is no difference in the ways individuals are informed, this value is unique both *a priori* and *a posteriori* and it is the same even with uncertainty about the future state of nature. To get a feeling of how messy it becomes as soon as differences in information arise, suffice it to say that two individuals are informed about the future states of nature by observing beforehand private signals that are different: the private signal y^i, observed by i, is a random variable correlated with the state of nature to come and the private signal y^l, observed by individual l, is a *different* random variable correlated (in a different way) with the state of nature to come. Let us call a "configuration of private signals" (thereafter CPS) the simultaneous realization of values for the signals respectively observed by the N individuals in the economy. A CSP is what is usually termed "relevant information about the fundamental", *i.e.* about h or Q_h.

A posteriori the market may be cleared in as many different ways as there are different CPS, producing as many values of the equilibrium price v. The random variable v is conditioned on the future state of nature from which the CPS originates.

Three more definitions are needed:

- the $\pi(Q_h)$ are the *a priori beliefs*, held before the private signals are observed, we suppose that they are common to all the individuals;

- $\pi_i(y^i_{k_i} \mid Q_h)$ is the probability of state h sending to individual i the value $y^i_{k_i}$ of his private signal y^i. Here we suppose that the random variable y^i takes a *finite* number of possible values $y^i_1, \ldots, y^i_{k_i}, \ldots, y^i_{K_i}$.

- the current CPS is a vector written $\overrightarrow{Y}_\mu$, with components $y^i_{k_i}$.

In that kind of models the two basic problems are:

- the precise formulation of the rational expectation hypothesis;

- the actual computing of the possible equilibrium values v_m and of their probabilities, the usual approach being through a fixed point.

A very simple example will illustrate

- the possibility of a *direct* computation once the relevant information has been *put in common*;

- the concept of "full revelation" (of the relevant information) by the equilibrium price.

3 An introductory example

Let us consider a very special case in the spirit of what has just been said:

- the number of states of nature is $H = 16$, with equal *a priori* probabilities;

- the number of individuals is $N = 3$ ($i \in \{a, b, c\}$).

Individual i maximise a Von Neumann utility with the elementary utility

$$u_i\left(W_i^+\right) = Log\left(W_i^+\right). \tag{4}$$

Every individual observes only his *private* signal *without noise.* Signal x, observed by a, may take either the value x_1 or the value x_2. The same definitions hold for y, y_1, y_2 (observed by b) and for z, z_1, z_2 (observed by c). A matrix is a good representation the three different "information structures":

states

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	x_1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
	x_2	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
private signals	y_1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
	y_2	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
	z_1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
	z_2	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1

It shows, for instance, that state 6 sends *simultaneously*

- towards a the value x_2 of signal x;

- towards b the value y_1 of signal y;

- towards c the value z_2 of signal z;

or that, after observing the value z_1 of signal z, individual c knows that the forthcoming state of nature belongs to the set $\{1, 2, 3, 4, 9, 10, 11, 12\}$.

There are eight CPS: $\overrightarrow{Y}_1 = \{x_1, y_1, z_1\}, \ldots, \overrightarrow{Y}_8 = \{x_2, y_2, z_2\}$.

The last data we need are

- Q_h, with $h \in \{1, \ldots, 16\}$, the possible amounts of numeraire good to be shared *a posteriori* between the stockholders:

- the *a priori* endowments of the individuals, in good (ω^i) and in stocks (θ_i^-).

The CH gathers the individual demand functions ($\theta_i = \delta_i(v)$ with $i \in \{a, b, c\}$) and computes the equilibrium value of the enterprise by solving the equation

$$\sum_i \delta_i(v) = 0. \tag{5}$$

Actually, from each individual may come two different demand functions, respectively associated with the two possible values of the signal he may observe. Eight possible cases are repectively associated to the eight different CPS. For each of them we have :

- a triplet of individual demand functions, one from each individual (according to his realized signal);

- a solution of equation () by the the CH;

- the posting of the computed equilibrium value v.

Those very events are rationally expected so that, generically, eight rationally expected equilibrium values of v are associated one-to-one with CPS $\overrightarrow{Y}_1, \ldots, \overrightarrow{Y}_m \ldots, \overrightarrow{Y}_8$, we may call them $v_1, \ldots, v_m, \ldots, v_8$.

The set $\{v_1, \ldots, v_8\}$ is the union of the definition sets of the six different demand functions (two for each individual).

Let us consider individual i who has just observed one value for his private signal and who is building and plotting, one point after the other, the associated demand function he will hand over to the CH. Conditionally on each v_m (actually on only the four of them conditioned on the value of the private signal he has just observed), he computes the value of the net proportion θ_i which maximize his expected utility. The computation for v_m rests on $\overrightarrow{Y}_m$ being given and providing the probabilities in equation (3). Simultaneously, for the same value v_m, all the other individuals are doing the same, that is using as a data the same $\overrightarrow{Y}_m$. To work on v_m gives them access to a common information (a supersignal) which is all the information available about the fundamental, hence the catch phrase "the equilibrium price fully reveals the relevant information".

The mapping between the rationally expected equilibrium values of v and the CPS being one-to-one, each v_m may be considered as following from all the agents "putting in common" all their private signals and using that supersignal (actually a CPS) to compute their demands. Finally, for the builder of the model, who wants to solve it, or for an individual who, according to the usual interpretation of the rational expectations hypothesis, "knows the model", a *direct* computation of all the v_m consists in running eight times in a row the same algorithm:

- pick a CPS;

- suppose it is known by all the individuals;

- update their (common) *a priori* beliefs by using this supersignal, according to Bayes rule;

- compute the equilibrium value of v.

So far three main ideas has been unearthed:

- the reason why the equilibrium price reveals some information is that an individual demand function is a list of proportions respectively *conditional* on the *rationally expected* equilibrium values of v;

- the equilibrium price is *fully* revealing because the mapping between the equilibrium values of v and ths CPS is *one-to-one*;

- the direct computation of the equilibrium values of v and of their probabilities rests on putting in common, one by one, all the CPS.

In our example, only two possible states of nature are associated with each CPS and their *a posteriori* probabilities are equal. For instance, the supersignal $\overrightarrow{Y_6} = \{x_2, y_1, z_2\}$ tells us that only states 6 and 14 are still possibles (with probabilities $\frac{1}{2}$ and $\frac{1}{2}$). We call $Q_f(m)$ and $Q_g(m)$ (with $Q_f(m) < Q_g(m)$) the two possible values of the profit after CPS $\overrightarrow{Y}_m$ has happened.

For each CPS put in common, the system of markets is complete and we can use the representative individual trick to compute directly the equilibrium value of v (cf. relation (25) in appendix A):

$$v_m = \frac{\Omega \left[Q_f(m) + Q_g(m)\right] + 2Q_f(m)\, Q_g(m)}{2\Omega + Q_f(m) + Q_g(m)} \tag{6}$$

$$\text{with } \ \Omega = \sum_i \omega^i.$$

Simple numerical values ($Q_h = h$ with $h \in \{1, \ldots, 16\}$, $\omega^a = 3$, $\omega^b = 2$ and $\omega^c = 5$) show how to make use of that formula to fill the first five columns in the table

CSP	m	$Q_f(m)$	$Q_g(m)$	v_m	$\delta_a^2(v_m)$	$\delta_b^1(v_m)$	$\delta_c^2(v_m)$
x_1, y_1, z_1	1	1	9	3,93		-0.0359	
x_2, y_1, z_1	2	2	10	5,00	-0.1333	-0.0333	
x_1, y_2, z_1	3	3	11	6,06			
x_2, y_2, z_1	4	4	12	7,11	-0.1169		
x_1, y_1, z_2	5	5	13	8,16		-0.0275	0.1377
x_2, y_1, z_2	6	6	14	9,20	-0.1042	-0.0260	0.1302
x_1, y_2, z_2	7	7	15	10,24			0.1235
x_2, y_2, z_2	8	8	16	11,27	-0.0940		0.1175

Here we have a very simple (and partial) "explanation" of the *volatility* of v.

Once the direct computation of the equilibrium values of v has been completed, we need numerical values for θ_a^-, θ_b^- and θ_c^- to build the individual demand functions to be forwarded to the CH. Each individual holds "personal rational expectations" conditional on the value of his private signal .For instance, individual a forwards

- a list of four proportions θ_a^1 respectively associated to v_1, v_3, v_5 and v_7, after he has just observed x_1;

- a list of four proportions θ_a^2 respectively associated to v_2, v_4, v_6 and v_8,.after he has just observed x_2.

Each of those $\delta_a^k(v_m)$, with $k \in \{1,2\}$, comes from the maximisatione the expected utility:

$$U_a(v_m) = \frac{1}{2}Log\left\{\omega^a + \theta_a^- v_m + \theta_a^+ \left[Q_f(m) - v_m\right]\right\} + \frac{1}{2}Log\left\{\omega^a + \theta_a^- v_m + \theta_a^+ \left[Q_g(m) - v_m\right]\right\},$$

which yields (relation (23) in appendix A)

$$\theta_a^+ = \frac{2v_m - \left[Q_f\left(m\right) + Q_g\left(m\right)\right]}{2\left[Q_f\left(m\right) - v_m\right]\left[Q_g\left(m\right) - v_m\right]}\left(\omega_a + \theta_a^- v_m\right) \tag{7}$$

The same approach applies to b and c. Once CPS $\overrightarrow{Y}_m$ has happened, the CH gets three individual demand functions, each of them being defined for only four values of v. There is only one value v_m for which all three functions are defined and $\sum_i \delta_i(v_m) = 0$, it is the unique equilibrium value of the price associated to $\overrightarrow{Y}_m$. In the simple case $\theta_a^- = \frac{1}{2}$, $\theta_b^- = \theta_c^- = \frac{1}{4}$ and with CPS $\overrightarrow{Y}_6$, the three individual demand functions appear as the last three columns in the table.

4 Scattered information and rational expectations

From now on, we will deal with *scattered information* proper. There is a perfect symetry in the way different people get their (*noisy*) signals: conditionally on any h, those random variables are i.i.d. The set of the possible values taken by the observed private signal is the same for all individuals ($y^i \in \{y_1, .., y_k, .., y_K\}$, $\forall i$) and there is a unique set of conditional probabilities $\pi(y_k \,|Q_h\,)$ with $k \in \{1, \ldots, K\}$ and $h \in \{1, \ldots, H\}$. The individuals have the same *information structures*, but, for a given state of nature, they still may observe different values of the signal. Actually, our introductory example was a case of scattered information but *without noise*: the values of probabilities $\pi(y_k \,|Q_h\,)$ could only be either zero or one.

Index i in the writing of probability $\pi_i(v_m, Q_h)$ in equation (3) reminds us of the fact that, when he builds his individual demand function, individual i is taking into account the value of y^i he has just observed (it is the only information available to him). More precisely, he has already revised his *a priori* beliefs according to Bayes rule. The *a posteriori* probability can be rewritten

$$\pi_i(v_m, Q_h) = \pi_i(v_m \, |Q_h\,)\pi_i(Q_h). \tag{8}$$

The result of the bayesian updating is

$$\pi_i(Q_h) = \pi(Q_h \mid y^i = y_k) = \frac{\pi(Q_h)\pi(y_k \mid Q_h)}{\sum\limits_{\alpha} \pi(Q_\alpha)\pi(y_k \mid Q_\alpha)}, \tag{9}$$

where the $\pi(Q_a)$, with $\alpha \in \{1, \ldots, h, \ldots, H\}$, are common *a priori* beliefs.

Conditional probabilities $\pi_i(v_m \, |Q_h\,)$ derive from the rational expectations hypothesis; as usual that means that

- *a prior*i, all individuals associates the possible equilibrium values of the price to the states of nature *in the same way* (index i must be deleted);

- this is the *true* equilibrium association actually solving the model.

Finally relation (8) must be written

$$\pi_i(v_m, Q_h) = \pi(v_m \, |Q_h\,)\pi(Q_h \, \big| y^i\,). \tag{10}$$

The successive steps of a closed loop reasoning mimicking the search for a fixed point are:

- all individuals have the same *priori* beliefs $\pi(Q_h)$ about the fundamental;

- after observing the value y_k of his private signal y^i, individual i revises his beliefs according to (9);

- all individuals know the true possible equilibrium values v_m and their conditional probabilities $\pi(v_m \, |Q_h\,)$;

- for each value v_m of the equilibrium price, individual i computes the value of $\theta_i^+ - \theta_i^-$ maximizing

$$\sum_h \pi(v_m \mid Q_h)\pi(Q_h \mid y_k)u_i \left[\omega_i + (\theta_i^- - \theta_i^+)v_m + \theta_i^+ Q_h\right]; \tag{11}$$

- the list of couples (θ_{im}, v_m), computed that way and depending on the observed value of the private signal y^i, is the individual demand function sent to the CH;

- the realization of a CPS $\overrightarrow{Y}_\mu$, the components of which are the observed values of the private signals y^i, induces the CH to compute a value v_μ for the equilibrium price, by solving an equation (5)

- since generically the mapping betwween CPS and equilibrium values of v is one-to-one we can write

$$\pi(v_m) = \pi\left(\overrightarrow{Y}_m\right), \tag{12}$$

$$\pi(v_m \mid Q_h) = \pi(\overrightarrow{Y}_m \mid Q_h); \tag{13}$$

- the loop is closed by the fact that the probabilities $\pi(v_m \mid Q_h)$ are actually "the true probabilities": rational expectations are self-fulfilling.

We have oulined what might be an iterated search for a fixed point in a complicated functional space. Fortunately, our introductory example has illustrated a much easier direct computation of the v_m and their probabilities. Here again the principle is:

- take, one after the other, each CPS;
- suppose it is known by all the individuals;
- update their (common) *a priori* beliefs by using this supersignal, according to Bayes rule;
- compute the equilibrium value of v and its probabilities (12) and (13).

An other simple example will help:

- there are two states of nature ($h \in \{f, g\}$);
- the *a posteriori* profit shared between individuals is either $Q_f = Q > 0$ or $Q_g = 0$;
- the common *a priori* beliefs are $\pi(Q_f) = \pi(Q_g) = \frac{1}{2}$;
- the value of the private signal observed by any individual is either x or z, with conditional probabilities

$$\pi(x \,|f\,) = \pi(z \,|g\,) = q,$$

$$\pi(x \,|g\,) = \pi(z \,|f\,) = 1 - q, \tag{14}$$

$$\text{with } \frac{1}{2} \leq q \leq 1;.$$

- the private signals are i.i.d. random variables.

Parameter q might be considered a measure of the *quality* of the signal: with $q = \frac{1}{2}$, one does not learn anything from observing a private signal, on the other hand, with $q = 1$ information is perfect. Nevertheless,

$$\lambda = \frac{2q - 1}{1 - q} \tag{15}$$

might be a better measure by allowing us to introduce, in a very simple way, "the cost of acquiring information": the "production function"

$$A = \alpha\lambda, \tag{16}$$

with $\alpha > 0$ and decreasing return, is a nice way to do so. It models the fundamental idea that one does not invest in becoming "better (or more) informed" by buying a bigger piece of a mysterious commodity called information (the notorious I_t, "quantity of information available at date t"), but by gaining access to a better signal, *i.e.* by becoming able to observe a random variable more closely correlated with the random variable of interest.

We expect CPS $\overrightarrow{Y}$ to be a vector with N components, each of them being either x or z (or being either 1 or 0).

Again we suppose that individuals have the same elementary utility function $u_i\left(W_i^+\right) = Log\left(W_i^+\right)$ so that

- to compute directly (*i.e.* after making the CPS publicly known) the possibles equilibrium values of v, we can use the representative individual trick;
- the only supersignal we have to make publicly known is the *total number* of individuals who have observed x. Let us call it X , with $X \in \{0, \ldots, N\}$. The number X is what statisticians call *a sufficient statistics* for the sample of observed private signals.

Since X is acting as CPS and the mapping between CPS and equilibrium values of v is one-to-one, we use X as an index for those values and write them as (relation (30) in appendix B)

$$v_X = \frac{Q}{1 + \frac{\Omega + Q}{\Omega}\left(\frac{q}{1-q}\right)^{N-2X}}, \tag{17}$$

or, because of (15),

$$v_X = \frac{Q}{1 + \left(1 + \frac{Q}{\Omega}\right)(1+\lambda)^{N-2X}}. \tag{18}$$

These relations embody a trivial "stylised fact": given the total endowment Ω and the common quality q of the private signals, the equilibrium value of v is higher if more individuals observe the propitious value of the private signal, that turns them into "optimists".

5 Partial revelation

In the preceding example, endowments ω^i (or at least their sum Ω) were nonrandom. It is natural to try to draw a parallel between uncertainties introduced on them and uncertainties on the private signals y^i. To complete a perfect parallel, we suppose that the set of possible values taken by individual endowments in numeraire good is the same for all individuals ($\omega^i \in \{\omega_1, .., \omega_j, .., y_J\}$, $\forall i$) and that there is a unique set of probabilities $\pi(\omega_j)$. The private signal, conditioning the choice by individual i of the demand function to be forwarded to the CH, is now made of two parts: the privately observed value of y^i and the privately observed value of ω^i. That calls for slight changes in terminology. The value of the 2-tuple (y^i, ω^i) is the "type" of individual i. KJ different individual types are possible and a combination of types, one for each individual, is a "configuration of individual types" (hereafter CIT). The most general CIT is a N-component vector $\overrightarrow{T_\tau}$ with component i beeing the (index of the) type of individual i.

Sticking with the general framework of the preceding example we see that

- the summary (sufficient statistics) form of the CIT is the couple (X, Ω);
- it is still possible to compute direcly the rationaly expected equilibrium values of v. What is now put in common (the "common supersignal") is the summary CIT instead of the summary CPS;
- relations (17) and (18) hold but with Ω being a random variable;
- as far as the trivial stylised fact is concerned, there seems to be an identification problem: does a medium observed value of v arises from many "optimists" being "poors" (recipients of small values of ω^i) or from many "poors" being "optimists"?

This last point is the important one. As long as we are in a discrete framework, with Ω being a discrete random variable, the mapping between the equilibrium values of v and the CIT is generically one-to-one, *i.e.*the function $v = V(X, \Omega)$ defined by (30) can be inverted. Being the relevant information, the X_m part of the CIT (X_m, Ω_m) is "fully revealed" by the value v_m of the equilibrium price. As soon as Ω becomes a continuous random variable, v too becomes a continuous random variable but, above all, several 2-uples (X, Ω) solve equation (30) for a given value of v. That value does not *fully* reveal any more, but that does not keep us from computing the probabily distribution of v and of v conditioned on any value of Q_h. Nothing changes in the method for computating directly the rationally expected values of the equilibrium price. The only difficulty arises in the computing of the probability distributions of v and of v conditioned on any value of Q_h. Now we deals with continuous random variables: instead of relations (12) and (13) we must handle probability density functions $\varphi(v)$ and $\varphi_h(v \mid Q_h)$. The probability density $\varphi(v)$ is given by formula (32) in Appendix B

Nevertheless the two pillars of our approach still hold:

- putting in common all the CIT yields the equilibrium values of v and their probabilities;

- individual demand functions are computed, one value after the other, with "personal" rational expectations (respectively conditioned on the observed values of the private signals).

Actually, we do not miss the one-to-one mapping between v and the X component of the CIT. There is nothing inefficient, "informationally" or otherwise, in the working of the market: it does its best to match supply and demand without squandering.

Once we have introduced the ω^i component of individual i's type, it is natural to go on adding new random characteristics of individuals. Some of them could be "shocks", for instance shocks on the risk aversion: if we had used utility functions $\frac{(W_i)^{1-\gamma^i}}{1-\gamma^i}$, γ^i could has been considered as a random variable in the same way as ω^i. The purest shocks (or noises) are random variables ε^i (with $\sum \varepsilon^i = 0$) directly added to the individual demands. Endowments θ_i^- (with $\sum \theta_i^- = 1$) and qualities λ^i (cf. definition (15)) of the private signals are our last examples.

Two complications arise, as in the case of random ω^i:

- as soon as any of the "new noise" (ω^i, γ^i, ε^i, etc.) is a continuous random variable, it drawns the "relevant information" (*i.e.* the CPS inside the CIT);

- in more general models, especially those without a representative individual, it might be impossible to find sufficient statistics for components of the CIT. The dimension of the CIT greatly increase, but it is still put in common to be used as a supersignal in the direct computation of the probability densities of v.

The most interesting improvement of our basic model (with ω^i being the only "new" noise) consists in making *endogeneous* the quality λ^i of the private signal observed by individual i: *a priori* he may use part of his wealth W_i^- to invest in "the costly gathering of an information" according to the production function (16).

6 Grossman-Stiglitz revisited

Let us suppose that endowments ω^i are the only new noises and that qualities λ^i are endogeneous, the other hypotheses being the same as for the model in section 4; this is the closest we can get to Grossman-Stiglitz [1980]. Here, the proper approach to understand wether individual i has an incentive to invest in the costly gathering of an information, bound to be stolen by an "informationally efficient market", is modelling the situation as a noncooperative games. A two-by-two example should suffice to illustrate the meaning and the limits of the Grossman-Stiglitz paradox.

Here again two individuals are a and b. Their endowments ω^a and ω^b are i.i.d. continuous random variables (their being uniformly distributed on $[\underline{\omega}, \overline{\omega}]$ would be convenient). As far as the costly gathering of information is concerned, both individuals have the same production function (16) and face the same alternative: either to invest $\underline{A}$ (low value) to get quality $\underline{\lambda}= \dfrac{\underline{A}}{\alpha}$ or to invest $\overline{A}$ (high value) to get quality $\overline{\lambda} = \dfrac{\overline{A}}{\alpha}$. The opportunity of introducing the new noise α^i (shock on individual i's "production function") is worth mentioning, of course we will not seize it.

Piling up new noises fogs the "explanation" of the observed bayesian Nash equilibrium: we can't be sure that a high equilibrium value of v proceeds from individuals having observed the good signals they paid for. If individual efforts and their results are covered up, the Grossman-Stiglitz result ceases to be clear-cut.

7 Conclusion: information revealing volumes?

The trick of the CH is a much better approach than the walrasian auctioneer to solve the paradox of the price conveying information *before* being set at its equilibrium value. The revelation of information rests on the fact that each individual *conditions* the proportion he asks for on the value of the price, so that, when the revelation by the price is incomplete, it is natural to look for other economic variables to assist the equilibrium price. A candidate must be

- unknown (but rationally expected) by individuals until it is observed or computed by the CH;

- observable on real life markets, so that it could be announced by an auctioneer together with the price.

The total endowment Ω is too obvious a choice. Since it is not even computed by the CH, it won't do the trick unless we suppose that the CH is some kind of national statistical office observing the public signal Ω ahead of the individuals.

More economically meaningful and more tightly linked to the equilibrium would be a statistics of the traded "volumes" (*i.e.* proportions). Since the CH observes *net* individual demands, the aggregate $\sum_i \theta_i$, being zero, is useless. We are left with other statistics of the sample $\{\theta_1, \dots, \theta_i, \dots, \theta_N\}$, for instance the variance or $\dfrac{1}{N} \sum_i |\theta_i|$. The problem is: can any of them be a public signal improving the bayesian update of each individual's beliefs?

The demand function transmitted by individual i to the CH has been written $\theta_i^+ = \delta_i(v)$, actually it comes from a three-parameter family and should be written $\theta_i^+ = \delta_i^*(\theta_i^-, \omega^i, y^i, v)$. We may imagine that, before getting his endowments θ_i^- and ω^i (ω^i at least is random) and before observing his private signal y^i, individual i builds the complete set of demand functions (*i.e.* for all combinations of values for y^i, ω^i and possibly θ_i^-), then, once he knows the values taken by ω^i and y^i, he picks out the relevant function and forwards it to the CH. The results of all those choices (one for each value of i) are observed by the CH to solve

$$\sum_i \delta_i^*(\theta_i^-, \omega^i, y^i, v) = 0. \tag{19}$$

By looking at the demand function transmitted by individual i, the CH can't figure out the values taken by ω^i and by y^i, but since (19), together with $\sum_i \theta_i^- = 1$, holds for the equilibrium value of v, computed and posted by the CH, there is, on the set of possibles values of the CIT, a restriction of the same kind as the one produced, for individual i, by the observation of his private signal and by the information (partially) revealed by the price. Is it possible to find a statistics of the sample of volumes (*i.e.* proportions) signaling that new restriction, so that individuals are in a position to condition their demand on the equilibrium value of that statistics?

8 Appendix A

Let us model the simplest exchange economy, with a walrasian stock market to trade the numeraire good for the stocks of the only enterprise. Individuals are not differently informed, but they may hold different *a priori* beliefs.

Within the *a priori*-*a posteriori* time framework, the economy consists of

- two states of nature ($h \in \{f, g\}$);

- the *a posteriori* profit Q_h, conditioned on the state of nature and shared by the entreprise between its stockholders;

- N individuals ($i \in \{1, \ldots, N\}$).

A priori, individual i is endowed with an amount ω^i of numeraire good and a proportion θ_i^- of the enterprise stocks; *a posteriori*, once the transactions have been settled on the stock market, the proportion held becomes θ_i^+. Individual i maximize the Von Neumann utility

$$U_i = \sum_h \pi_i(h) Log\left[\omega^i + \left(\theta_i^+ - \theta_i^-\right) v + \theta_i^+ Q_h\right], \tag{20}$$

where the probabilities $\pi_i(h)$ are individual i's *a priori* beliefs about the states.

The first order condition $\dfrac{\partial U_i}{\partial \theta_i^+} = 0$ can be written

$$\frac{\omega^i + \theta_i^- v + \theta_i^+(Q_f - v)}{\pi_i(f)(Q_f - v)} + \frac{\omega^i + \theta_i^- v + \theta_i^+(Q_g - v)}{\pi_i(g)(Q_g - v)} = 0, \tag{21}$$

or

$$\theta_i^+ \left[\frac{1}{\pi_i(f)} + \frac{1}{\pi_i(g)}\right] = -\left(\omega^i + \theta_i^- v\right)\left[\frac{1}{\pi_i(f)(Q_f - v)} + \frac{1}{\pi_i(g)(Q_g - v)}\right], \tag{22}$$

hence

$$\theta_i^+ = \frac{v - [\pi_i(f)Q_f + \pi_i(g)Q_g]}{(Q_f - v)\,(Q_g - v)} \left(\omega^i + \theta_i^- v\right). \tag{23}$$

Let us assume that all individuals hold the *same beliefs* ($\pi_i(f) = \pi_f$ and $\pi_i(g) = \pi_g$, $\forall i$).

Owing to $\sum_i \omega^i = \Omega$ and $\sum_i \theta_i^- = 1$, the equilibrium condition $\sum_i \theta_i^+ = 1$ yields

$$(Q_f - v)\,(Q_g - v) = (\Omega + v)\,[v - (\pi_f Q_f + \pi_g Q_g)]\,, \tag{24}$$

hence

$$v = \frac{\Omega\,(\pi_f Q_f + \pi_g Q_g) + Q_f Q_g}{\Omega + (1 - \pi_f)Q_f + (1 - \pi_g)Q_g}. \tag{25}$$

Of course, we could get that relation by the much simpler method of introducing the *représentative individual*, which simply amounts to replacing ω^i by Ω, θ_i^- by 1 and θ_i^+ by 1 in (21).

Still more special, the hypothesis $\pi_f = \pi_g = \frac{1}{2}$, leads to

$$\theta_i^+ = \frac{2v - (Q_f + Q_g)}{2(Q_f - v)\,(Q_g - v)} \left(\omega^i + \theta_i^- v\right), \tag{26}$$

and (25) becomes

$$v = \frac{\Omega\,(Q_f + Q_g) + 2Q_f Q_g}{2\Omega + Q_f + Q_g}. \tag{27}$$

9 Appendix B

Let us model the simple case of two states, f (with $Q_f = Q$) and g (with $Q_g = 0$), and a two-value *noisy* signal with "quality" q.

$$q = \pi(x\,|f)$$

$$\begin{array}{ccccccccc}
x & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & Q_f \\
& \nwarrow & & & & & & \swarrow & \\
& & \nwarrow & & & & \swarrow & & \\
& & & \nwarrow & & \swarrow & 1-q & & \\
& & & & \begin{smallmatrix}\nwarrow & \swarrow \\ \swarrow & \nwarrow\end{smallmatrix} & & & & \\
& & & \swarrow & & \nwarrow & 1-q & & \\
& & \swarrow & & & & \nwarrow & & \\
& \swarrow & & & & & & \nwarrow & \\
z & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & Q_g \\
& & & & q=\pi(z\,|g) & & & &
\end{array}$$

The number of individuals is N. All of them have the same logarithmic elementary utility, so that we can say that the number X (with $X \in \{0, \ldots, N\}$) of those who observe the private signal x is the CPS.

The probabilities of the CPS X being brought about by the states of nature are

$$\pi(X \mid Q) = \binom{N}{X}.q^X.(1-q)^{N-X} \quad \text{and} \quad \pi(X \mid 0) = \binom{N}{X}(1-q)^X.q^{N-X}. \quad (28)$$

Since the observed private signals are *put in common*, eveybody updates in the same way his (common) *a priori* beliefs, so that the *a posteriori* beliefs are the same:

$$\pi(Q \mid X) = \frac{\pi(Q).\pi(X \mid Q)}{\pi(Q).\pi(X \mid Q) + \pi(0).\pi(X \mid 0)}, \quad (29)$$

$$\pi(0 \mid X) = \frac{\pi(0).\pi(X \mid 0)}{\pi(Q).\pi(X \mid Q) + \pi(0).\pi(X \mid 0)}.$$

The logarithmic utilities and the complete markets allow us to introduce a representative individual and to rewrite () as

$$Q - v = \frac{\pi(0 \mid X).(\Omega + Q)}{\pi(Q \mid X).\Omega}.v,$$

or :

$$Q - v = \frac{\pi(0)}{\pi(Q)}.\frac{\binom{N}{X}.(1-q)^X.q^{N-X}}{\binom{N}{X}.q^X.(1-q)^{N-X}}.\frac{(\Omega + Q)}{\Omega}.v,$$

hence :

$$Q - v = \left(\frac{q}{1-q}\right)^{N-2X}.\frac{\Omega + Q}{\Omega}.v,$$

and finally

$$v = V_X(\Omega) = \frac{Q}{1 + \left(1 + \frac{Q}{\Omega}.\right)\left(\frac{q}{1-q}\right)^{N-2X}}. \tag{30}$$

We will need the inverse function

$$\Omega = V_X^{-1}(v) = \frac{Q\left(\frac{q}{1-q}\right)^{N-2X}}{\frac{Q}{v} - \left(\frac{q}{1-q}\right)^{N-2X} - 1}. \tag{31}$$

If Ω is a (continuous) random variable, let $\Psi(\Omega)$ be its repartition function. The repartition function of v (conditioned on X) is

$$\Phi_X(v) = \Psi\left[V_X^{-1}(v)\right]$$

and the probability density $\varphi_X(v)$ is its derivative.

Finally, the probability density of v is given by

$$\varphi(v) = \sum_X \pi(X)\phi_X(v). \tag{32}$$

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