Herd behavior and aggregate fluctuations in financial markets [1]

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Abstract

We present a simple model of a stock market where a random communication structure between agents generically gives rise to a heavy tails in the distribution of stock price variations in the form of an exponentially truncated power-law, similar to distributions observed in recent empirical studies of high frequency market data. Our model provides a link between two well-known market phenomena: the heavy tails observed in the distribution of stock market returns on one hand and 'herding' behavior in financial markets on the other hand. In particular, our study suggests a relation between the excess kurtosis observed in asset returns, the market order flow and the tendency of market participants to imitate each other.

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Empirical studies of the fluctuations in the price of various financial assets have shown that distributions of stock returns and stock price changes have fat tails that deviate from the Gaussian distribution [38, 39, 12, 27, 42, 14, 43] especially for intraday time scales [14]. These fat tails, characterized by a significant excess kurtosis, persist even after accounting for heteroskedasticity in the data [9]. The heavy tails observed in these distributions correspond to large fluctuations in prices, "bursts" of volatility which are difficult to explain only in terms of variations in fundamental economic variables [47].

The fact that significative fluctuations in prices are not necesarily related to the arrival of information [16] or to variations in fundamental economic variables [47] leads to think the high variability present in stock market returns may correspond to collective phenomena such as crowd effects or "herd" behavior.

Although herding in financial markets is by now relatively well documented empirically, there have been few theoretical studies on the implications of herding and imitation for the statistical properties of market demand and price fluctuations. In particular some questions one would like to answer is: how does the presence of herding modify the distribution of returns? What are the implications of herding for relations between market variables such as order flow and price variability? These are some of the questions which have motivated our study.

The aim of the present study is to examine, in the framework of a simple model, how the existence of herd behavior among market participants may generically lead to large fluctuations in the aggregate excess demand, described by a heavy-tailed non-Gaussian distribution. Furthermore we explore how empirically measurable quantities such as the excess kurtosis of returns and the average order flow may be related to each other in the context of our model. Our approach provides a quantitative link between the two issues discussed above: the *heavy tails* observed in the distribution of stock market returns on one hand and the *herd* behavior observed in financial markets on the other hand.

The article is divided into four sections. Section 1 reviews well known empirical facts about the heavy-tailed nature of the distribution of stock returns and various models proposed to account for it. Section 2 presents previous empirical and theoretical work on herding and imitation in financial markets in relation to the present study. Section 3 discusses the statistical properties of excess demand resulting from the aggregation of of a large number of random individual demands in a market. Section 4 defines our model and presents analytical results. Section 5 interprets the results in economic terms, compares them to empirical data and discusses possible extensions. Details of calculations are given in the appendices.

1 The heavy-tailed nature of asset return distributions

It is by now a well known fact that the distribution of returns of almost all financial assets -stocks, indexes and futures- exhibit a slow asymptotic decay that deviates from a normal distribution. This is quantitatively reflected in the excess kurtosis, defined as:

$$\kappa = \frac{\mu_4}{\sigma^4} - 3 \tag{1}$$

where μ_4 is the fourth central moment and σ the standard deviation of the returns. κ should be zero for a normal distribution but ranges between 2 and 50 for daily returns [12, 43] and is even higher for intraday data. Careful study of the tails of the distribution shows an exponential decay for most assets [14, 40].

Many statistical models have been put forth to account for the heavy tails observed in the distribution of asset returns. Well known examples are Mandelbrot's stable paretian hypothesis [38] the mixture of distributions hypothesis [13], and models based on conditional heteroskedasticity [18].

It is well known that in the presence of heteroskedasticity, the unconditional distribution of returns will have heavy tails. In most models based on heteroskedasticity, the process of return is assumed to be conditionally Gaussian : the shocks are "locally" Gaussian and the non-Gaussian character of the unconditional distribution is an effect of aggregation. It is obtained by superposing a large number of local Gaussian shocks. In this description, sudden movements in prices are interpreted as corresponding to a high value of conditional variance.

On one hand, it has been shown that although conditional heteroskedasticity does lead to fat-tails in unconditional distributions, ARCH-type models cannot fully account for the kurtosis of returns [29, 9]. On the other hand, from a theoretical point of view there is no *a priori* reason to postulate that returns are conditionally normal: although conditional normality is convenient for parameter estimation of the resulting model, non-normal conditional distributions possess the same qualitative features as for volatility clustering while accounting better for heavy tails. Gallant and Tauchen [23] report significant evidence of both conditional heteroskedasticity and conditional non-normality in the daily NYSE value-weighted index. Similarly, Engle and Gonzalez-Rivera [17] show that when a GARCH(1,1) model is used for the conditional variance of stock returns the conditional distribution has considerable kurtosis, especially for small firm stocks. Indeed several authors have proposed GARCH-type models with non-normal conditional distributions [9].

Stable distributions [38] offer an elegant alternative to heteroskedasticity for generating fat tails, with the advantage that they have a natural interpretation in terms of aggregation of a large number of individual contributions of agents to market fluctuations: indeed, stable distribution may be obtained as limit distributions of sums of independent or weakly dependent random variables, a property which is not shared by alternative models. Unfortunately, the infinite variance property of these distributions is not observed in empirical data: sample variances do not increase indefinitely with sample size but appear to stabilize at a certain value for large enough data sets. We will discuss stable distributions in more detail in Section 3.

A third approach, first advocated by Clark [13], is to model stock returns by a subordinated process, typically subordinated Brownian motion. This amounts to stipulating that through a "stochastic time change" one can transform the complicated dynamics of the price process into Brownian motion or some other simple process. It can be shown that, depending on the choice of the subordinator, one can obtain a wide range of distributions for the increments all of which possess heavy tails i.e. positive excess kurtosis. As a matter of fact, even stable distributions may be obtained as a subordinated Brownian motion. In the original approach of Clark [13], the subordinator was taken to be trading volume. Other choices which have been proposed are the number of trades [24] or other local measures of market activity. However, none of these choices for the subordinator lead to a normal distribution for the increments of the time-changed process, indicating that large fluctuations in price may not be completely explained by large fluctuations in trading volume or number of trades.

In short, although heteroskedasticity and time deformation partly explain the kurtosis of asset returns, they do not explain it quantitatively: even after accounting for these effects, one is left with an important residual kurtosis in the resulting transformed time series. Moreover, these approaches are not based on any particular model of the market phenomenon generating the data that they attempt to describe.

Recent works by Bak, Paczuski and Shubik [3] and Caldarelli, Marsili and Zhang [11] have tried to explain the heavy tailed nature of return distributions as an emergent property in a market where fundamentalist traders interact with noise traders. Bak, Paczuski and Shubik consider several types of trading rules and study the resulting statistical properties for the time series of asset prices in each case. Computer simulations of their model do seem to yield fat-tailed distributions for asset returns which at least qualitatively resemble empirical distributions of stock returns, showing that the appearance of fat-tailed distributions can be regarded as an emergent property in large markets. However, the model has two drawbacks: first, it is a fairly complicated model with many ingredients and parameters and it is difficult to see how each ingredient of the model affects the results obtained, which in turn diminishes its explanatory power. Second, the complexity of the model does not allow explicit calculations to be performed, preventing the model parameters to be compared with empirical values.

We present here an alternative approach which, by modeling the communication structure between market agents as a *random graph*, proposes a simple mechanism accounting for some non-trivial statistical properties of stock price fluctuations. Although much more rudimentary and containing fewer ingredients than the model proposed by Bak, Paczuski and Shubik, our model allows for analytic calculations to be performed, thus enabling us to interpret in economic terms the role of each of the parameters introduced. The basic intuition behind our approach is simple: interaction of market participants through imitation can lead to large fluctuations in aggregate demand, leading to heavy tails in the distribution of returns.

2 Herd behavior in financial markets

A number of recent studies have considered mimetic behavior as a possible explanation for the excessive volatility observed in financial markets [5, 47, 49]. The existence of herd behavior in speculative markets has been documented by a certain number of studies: Scharfstein and Stein [44] discuss evidence of herding in the behavior of fund managers, Grinblatt *et al.* [26] report herding in mutual fund behavior while Trueman [50] and Welch [51] show evidence for herding in the forecasts made by financial analysts.

On the theoretical side several studies have shown that, in a market with noise traders, herd behavior is not necessarily "irrational" in the sense that it may be compatible with optimizing behavior of the agents [48]. Other motivations may be invoked for explaining imitation in markets, such as "group pressure" [6, 44].

Various models of herd behavior have been considered in the literature, the most well known approach being that of Bannerjee [4, 5] and Bikhchandani *et al.* [6]. In these models, individuals attempt to infer a parameter from noisy observations and decisions of other agents, typically through a Bayesian procedure, giving rise to "information cascades" [6]. An important feature of these models is the sequential character of the dynamics: individuals make their decisions one at a time, taking into account the decisions of the individuals preceding them. The model therefore assumes a natural way of ordering the agents. This assumption seems unrealistic in the case of financial markets: orders from various market participants enter the market simultaneously and it is the interplay between different orders that determines aggregate market variables[2].

Non-sequential herding has been studied in a Bayesian setting by Orléan [41] in a framework inspired by the Ising model. Orléan considers imitation between agents in which any two agents have the same tendency to imitate each other. In terms of aggregate variables, this model leads either to a Gaussian distribution when the imitation is weak, or to a bimodal distribution with non-zero modes, which Orléan interprets as corresponding to collective market phenomena such as crashes or panics. In neither case does one obtain a heavy-tailed unimodal distribution centered at zero such as those observed for stock returns.

The approach proposed in this paper is different from both approaches described above. Our model is different from that in [4, 6]. in that herding is not sequential. The unrealistic nature of the results in [41] result from the fact that all agents are assumed to imitate each other to the same degree.

[2]Bikhchandani *et al* [6] do not consider their model as applicable to financial markets but for another reason: they remark that as the herd grows, the cost of joining it will also grow, discouraging new agents to join. This aspect, which is not taken into account by their model, is again unavoidable to the sequential character of herd formation.

We avoid this problem by considering the random formation of groups of agent who imitate each other but such that different groups of agents make independent decisions, which allows for a heterogeneous market structure. More specifically, our approach considers the interactions between agents as resulting from a *random* communication structure, as explained below.

3 Aggregation of random individual demands

Consider a stock market with N agents, labeled by an integer $1 \leq i \leq N$, trading in a single asset, whose price at time t will be denoted $x(t)$. During each time period, an agent may choose either to buy the stock, sell it or not to trade. The demand for stock of agent i is represented by a random variable ϕ_i, which can take the values 0, -1 or +1: a positive value of ϕ_i represents a "bull"- an agent willing to buy stock-, a negative value a "bear", eager to sell stock while $\phi_i = 0$ means that agent i does not trade during a given period. The random character of individual demands may be due either to heterogeneous preferences or to random resources of the agents, or both. Alternatively, it may result from the application by the agents of simple decision rules, each group of agents using a certain rule. However, in order to focus on the effect of herding, we do not explicitly model the decision process leading to the individual demands and model the result of the decision process as a random variable ϕ_i. In contrast with many binary choice models in the microeconomics literature, we allow for an agent to be inactive i.e. not to trade during a given time period t. This, as we shall see below, is important for deriving our results.

Let us consider for simplification that, during each time period, an agent may either trade one unit of the asset or remain inactive. The demand of the agent i is then represented by $\phi_i \in \{-1, 0, +1\}$, $\phi_i = -1$ representing a sell order. The aggregate excess demand for the asset at time t is therefore simply

$$D(t) \quad = \quad \sum_{i=1}^{N} \phi_i(t) \tag{2}$$

given the algebraic nature of the ϕ_i. The marginal distribution of agent is individual demand will is assumed to be symmetric :

$$P(\phi_i = +1) = P(\phi_i = -1) = a \qquad P(\phi_i = 0) = 1 - 2a$$

such that the average aggregate excess demand is zero i.e. the market is considered to fluctuate around equilibrium. A value of $a < 1/2$ allows for a finite fraction of agents not to trade during a given period.

We are concerned here with obtaining a result which could then be compared with actual market data and the short term excess demand is not an easily observable quantity. Also, most of the studies on the statistical properties of financial time series have been done on returns, log returns or price changes. We therefore need to relate the aggregate excess demand in a given period to the return or price change during that period. The aggregate excess demand has an impact on the price of the stock, causing it to rise if the excess demand is positive, to fall if it is negative. A common specification, which is compatible with standard *tatonnement* ideas, is to assume a proportionality between price change (or return) and excess demand:

$$\Delta x = x(t+1) - x(t) \;\; = \;\; \frac{1}{\lambda} \sum_{i=1}^{N} \phi_i(t) \tag{3}$$

where λ is a factor measuring the liquidity or, more precisely, the *market depth* [32]: it is the excess demand needed to move the price by one unit: it measure the sensitivity of price to fluctuations in demand. Eq. (3), emphasizes the price impact of the order flow as opposed to other possible causes for price fluctuations. Eq.(3) may be considered either in absolue terms with $x(t)$ being the price, or as representing *relative* variations of the price, $x(t)$ then being considered as the log of the price and its increment as the instantaneous return. The latter has the advantage of guaranteeing the positivity of the price but for short-run dynamics the two specifications do not differ substantially since the two quantities have the same empirical properties. A similar model for the price impact of *trades* has been considered by Hausman, Lo and MacKinlay [28]. Although in the long run economic factors other than short term excess demand may influence the evolution of the asset price, resulting in mean-reversion or more complex types of behavior, we focus here on the short-run behavior of prices, for example on intra-day time scales in the case of stock markets, so this approximation seems reasonable. The linear nature of this relation may also be questioned: indeed, some empirical studied seem to indicate that the price impact of trades may be non-linear [12, 32]. First, note that these studies deal with the price impact of trades and not of order flow (excess demand), which is much harder to measure. Results reported by Farmer and co-workers [21] based on the study of the price impact of blocks of orders of different sizes sent to the market seem to indicate a linear relationship for small price changes with nonlinearity arising when the size of blocks is increased. Moreover, if the one-period return Δx is a non-linear but smooth function $h(D)$ [3] of the excess demand, then a linearization of the inverse demand function h (a first order Taylor expansion in D) shows that Eq.(3) still holds for small fluctuations of the aggregate excess demand with $h'(0) = 1/\lambda$.

In order to evaluate the distribution of stock returns from Eq.(3), we need to know the *joint* distribution of the individual demands $(\phi_i(t))_{1 \leq i \leq N}$. Let us begin by considering the simplest case where individual demands ϕ_i of different agents are independent identically distributed random variables. We shall refer to this hypothesis as the "independent agents" hypothesis. In this case the joint distribution of the individual demands is simply the product of individual distributions and the price variation Δx is a sum of N *iid* random variables with finite variance. When the number of terms in Eq.(3) is large the central limit theorem applied to the sum in Eq.(3) tells us that the distribution of Δx is well approximated by a Gaussian distribution. Of course, this result still holds as long as the distribution of individual demands has finite variance.

This can be seen as a rationale for the frequent use of the normal distribution as a model for the distribution of stock returns: indeed, if the variation of market price is seen as the sum of a large number of independent or weakly dependent random effects, it is plausible that a Gaussian description should be a good one.

Unfortunately, empirical evidence tells us otherwise: the distributions both of asset returns [43, 12] and of asset price changes [38, 39, 14, 15] have been repeatedly shown to deviate significantly from the Gaussian distribution, exhibiting fat tails and excess kurtosis.

But the independent agent model is also capable of generating aggregate

[3] It is interesting to note that if $\Delta x = h(D)$, where h is an increasing function of D and if the individual demands $(\phi_i(t))$ are sequences of independent random variables (a somewhat extreme assumption), then it is easy to show that the overall wealth of all traders increases on average with time.

distributions with heavy tails: indeed, if one relaxes the assumption that the individual demands ϕ_i have a finite variance then under the hypothesis of independence (or weak dependence) of individual demands, the aggregate demand -and therefore the price change if we assume Eq.(3)- will have a stable (Pareto-Lévy) distribution. This is a possible interpretation for the stable-Paretian model proposed by Mandelbrot [38] for the heavy tails observed in the distribution of the increments of various market prices. The infinite variance of the ϕ_i then reflects the heterogeneity of the market, for example in terms of broad distribution of wealth of the participants as proposed by Levy & Solomon [37].

Mandelbrot's stable-Paretian hypothesis has been criticized for several reasons, one of them being that it predicts an infinite variance for stock returns which implies in practice that the sample variance will indefinitely increase with sample size, a property which is not observed in empirical data.

More precisely, a careful study of the tails of the distribution of increments for various financial assets shows [10, 14] that they have heavy tails with a finite variance. Many distributions verify these conditions [12]; a particular example proposed by the authors and others [14] is an exponentially truncated stable distribution the tails of the density then have the asymptotic form of an exponentially truncated power law:

$$p(\Delta x) \underset{|\Delta x| \to \infty}{\sim} \frac{C}{|\Delta x|^{1+\mu}} exp\,(-\frac{\Delta x}{\Delta x_0}) \tag{4}$$

The exponent μ is found to be close to 1.5 ($\mu \simeq 1.4 - 1.6$) for a wide variety of stocks and market indexes [10]. This asymptotic form allows for heavy tails (excess kurtosis) without implying infinite variance.

However, it is known the central limit theorem also holds for certain sequences of dependent variables: under various types of *mixing* conditions [7], which are mathematical formulations of the notion of "weak" dependence, aggregate variables will still be normally distributed. Therefore the non-Gaussian and more generally non-stable character of empirical distributions, be it excess demand or the stock returns, not only demonstrates the failure of the "independent agent" approach, but also shows that such an approach is not anywhere close to being a good approximation: the dependence between individual demands is an essential character of the market structure and may not be left out in the aggregation procedure, they cannot be assumed to be "weak" (in the sense of a mixing condition [7]) and *do* change the distribution of the resulting aggregate variable,

Indeed, the assumption that the outcomes of decisions of individual agents may be represented as independent random variables is highly unrealistic: such an assumption ignores an essential ingredient of market organization, namely the *interaction* and *communication* among agents.

In real markets, agents may form groups of various sizes which then may share information and act in coordination. In the context of a financial market, groups of traders may align their decisions and act in unison to buy or sell; a different interpretation of a "group" may be an investment fund corresponding to the wealth of several investors but managed by a single fund manager.

In order to capture such effects we need to introduce an additional ingredient, namely the communication structure between agents. One solution would be to specify a fixed trading group structure and then proceed to study the resulting aggregate fluctuations. Such an approach has two major drawbacks. First, a realistic market structure may require specifying a complicated structure of clusters and rendering the resulting model analytically intractable. More importantly, the resulting pattern of aggregate fluctuations will crucially depend on the specification of the market structure.

An alternative approach, suggested by Kirman [33], is to consider the market communication structure itself as stochastic. One way of generating a random market structure is to assume that market participants meet randomly and trades take place when an agent willing to buy meets and agent willing to sell. This procedure, called "random matching" by some authors [30], has been previously considered in the context of formation of trading groups by Ioannides [30] and in the context of a stock market model by Bak, Paczuski and Shubik [3]. Another way is to consider that market participants form groups or"clusters" through a random matching process but that no trading takes places inside a given group: instead, members of a given group adopt a common market strategy (for example, they decide to buy or sell or not to trade) and different groups may trade with each other through a centralized market process. In the context of a financial market, clusters may represent for example a group of investors participating in a mutual fund. This is the line we will follow in this paper.

4 Presentation of the model

More precisely, let us suppose that agents group together in coalitions or *clusters* and that, once a coalition has formed, all it's members coordinate their individual demands so that all individuals in a given cluster have the same belief regarding future movements of the asset price. In the framework described in the preceding section, we will consider that all agents belonging to a given cluster will have the same demand ϕ_i for the stock. In the context of a stock market, these clusters may correspond for example to mutual funds e.g. portfolios managed by the same fund manager or to herding among security analysts as in [50, 51]. The right hand side of the equation (3) may therefore be rewritten as a sum over clusters :

$$\Delta x = \frac{1}{\lambda}\sum_{\alpha=1}^{k} W_\alpha \phi_\alpha(t) \quad = \quad \frac{1}{\lambda}\sum_{\alpha=1}^{n_c} X_\alpha \tag{5}$$

where W_α is the size of cluster α, $\phi_\alpha(t)$ the (common) individual demand of agents belonging to the cluster α, n_c the number of clusters (coalitions) and $X_\alpha = \phi_\alpha W_\alpha$.

One may consider that coalitions are formed through binary links between agents, a link between two agents meaning that they undertake the same action on the market i.e. they both buy or sell stock. For any pair of agents i and j, let p_{ij} be the probability that i and j are linked together. Again, in order to simplify, we assume that $p_{ij} = p$ is independent of i and j: all links are equally probable. $(N-1)p$ then denotes the average number of agents a given agent is linked to. Since we are interested in studying the $N \to \infty$ limit, p should therefore be chosen in such a way that $(N-1)p$ has a finite limit. A natural choice is $p_{ij} = c/N$, any other choice verifying the above condition being asymptotically equivalent to this one. The distribution of coalition sizes in the market is thus completely specified by a single parameter, c, which represents the willingness of agents to align their actions: it can be interpreted as a coordination number, measuring the degree of clustering among agents.

Such a structure is known as a *random graph* in the mathematical literature [19, 8]: in terms of random graph theory, we consider agents as vertices of a random graph of size N, and the coalitions as connected components of the graph. Such an approach to communication in markets using random graphs was first suggested in the economics literature by Kirman [33] to study the properties of the core of a large economy. Random graphs have also been used in the context of multilateral matching in search problems by Ioannides [30]. A good review of the applications of random graph theory in economic modeling is given in [31].

The properties of large random graphs in the $N \to \infty$ limit were first studied by Erdös and Renyi [19]. An extensive review of mathematical results on random graphs is given in [8]. The main resullts of the combinatorial approach are given in Appendix 1. One can show [8] that for $c = 1$ the probability density for the cluster size distribution decreases asymptotically as a power law:

$$P(W) \underset{W\to\infty}{\sim} \frac{A}{W^{5/2}}$$

while for values of c close to and smaller than 1 ($0 < 1 - c << 1$), the cluster size distribution is cut off by an exponential tail:

$$P(W) \underset{W\to\infty}{\sim} \frac{A}{W^{5/2}} \exp(-\frac{(c-1)W}{W_0}) \tag{6}$$

For c=1, the distribution has an infinite variance while for $c < 1$ the variance becomes finite because of the exponential tail. In this case the average size of a coalition is of order $1/(1-c)$ and the average number of clusters is then of order $N(1-c/2)$.

Setting the coordination parameter c close to 1 means that each agent tends to establish a link with one other agent, which can be regarded as a reasonable assumption. This does not rule out the formation of large coalitions through successive binary links between agents but prevents a single agent from forming multiple links, as would be the case in a centralized communication structure where one agent (the "auctioneer") is linked to all the others. As argued by Kirman [33] the presence of a Walrasian auctioneer corresponds to such 'star-like', centralized, communication structures. We are thus excluding such a situation by construction: we are interested in a market where information is distributed and not centralized, which corresponds more closely to the situations encountered in real markets. More precisely, the local structure of the market may be characterized by the following result

[8]: in the limit $N \to \infty$, the number ν_i of neighbors of a given agent i is a Poisson random variable with parameter c:

$$P(\nu_i = \nu) \;=\; e^{-c}\frac{c^{\nu}}{\nu!} \tag{7}$$

A Walrasian auctioneer w would be connected to every other agent: $\nu_w = N-1$. The probability for having a Walrasian auctioneer is therefore given by $NP(\nu_w = N-1)$ which goes to zero when $N \to \infty$.

A given market cluster is characterized by its size W_α and its 'nature' i.e. whether the members are buyers or sellers. This is specified by a variable $\phi_\alpha \in \{-1, 0, 1\}$. It is reasonable to assume that W_α and ϕ_α are independent random variables: the size of a group does not influence its decision whether to buy or sell. The variable $X_\alpha = \phi_\alpha W_\alpha$ is then symmetrically distributed with a mass of $1-2a$ at the origin. Let

$$F(x) \;=\; P(X_\alpha \le x | X_\alpha \ne 0) \tag{8}$$

Then the distribution of X_α is given by

$$G(x) \;=\; P(X_\alpha \le x) = (1-2a)H(x) + 2aF(x) \tag{9}$$

where H is a unit step function at 0 (Heaviside function). We shall assume that F has a continuous density, f. f then decays asymptotically as in (6):

$$f(x) \underset{|x|\to\infty}{\sim} \frac{A}{|x|^{5/2}} e^{\frac{-(c-1)|x|}{W_0}} \tag{10}$$

The expression for the price variation Δx therefore reduces to a sum of n_c *iid* random variables $X_\alpha, \alpha = 1..n_c$ with heavy-tailed distributions as in (6):

$$\Delta x = \frac{1}{\lambda}\sum_{\alpha=1}^{n_c} X_\alpha$$

Since the probability density of X_α has a finite mass $1-2a$ at zero, only a fraction $2a$ of the terms in the sum (5) are non-zero; the number of non-zero terms in the sum is of order $2a\overline{n_c} \sim 2aN(1-c/2) = N_{order}(1-c/2)$ where

$N_{order} = 2aN$ is the average number of market participants who actively trade in the market during a given period. For example, N_{order} can be thought of as the number of orders received during the time period $[t, t+1]$ if we assume that different orders correspond to net demands, as defined above, of different clusters of agents. For a time period of, say, 15 minutes on a liquid market such as NYSE, $N_{order} = 100 - 1000$ is a typical order of magnitude.

The distribution of the price variation δx is then given by

$$P(\Delta x = x) \quad = \quad \sum_{k=1}^{N} P(n_c = k) \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} (2a)^j (1-2a)^{k-j} f^{\otimes j}(\lambda x) \quad (11)$$

where $\otimes$ denotes a convolution product, n_c being the number of clusters. The above equation enables us to calculate the moment generating functions $\mathcal{F}$ of the aggregate excess demand D in terms of $\tilde{f}$ (see Appendix 3 for details):

$$\mathcal{F}(z) \underset{N \to \infty}{\sim} \exp[N_{order}(1 - \frac{c}{2})(\tilde{f}(z) - 1)] \quad (12)$$

The moments of D (and those of Δx) may now be obtained through a Taylor expansion of Eq.(12) (see Appendix 4 for details). The calculation of the variance and the fourth moment yields:

$$\mu_2(D) \quad = \quad N_{order}(1 - \frac{c}{2})\mu_2(X_\alpha) \quad (13)$$

$$\mu_4(D) \quad = \quad N_{order}(1 - \frac{c}{2})\mu_4(X_\alpha) + 3N_{order}^2(1 - \frac{c}{2})^2\mu_2(X_\alpha)^2 \quad (14)$$

An interesting quantity is the kurtosis of the asset returns which, in our model, is equal to the kurtosis of excess demand $\kappa(D)$:

$$\kappa(D) = \frac{\mu_4(X_\alpha)}{N_{order}(1 - \frac{c}{2})\mu_2(X_\alpha)}$$

The moments $\mu_j(X_\alpha)$ may be obtained may be obtained by an expansion in 1/N where N is the number of agents in the market (see Appendix 2). Substituting their expression on the above formula yields the kurtosis $\kappa(D)$ as a function of c and the order flow:

$$\kappa(D) = \frac{2c+1}{N_{order}(1 - \frac{c}{2})A(c)(1-c)^3} \quad (15)$$

where $A(c)$ is a normalization constant with a value close to 1 defined in Appendix 2, tending to a finite limit as $c \to 1$. This relation may be interpreted as follows: a reduction in the volume of the order flow results in larger price fluctuations, characterized by a larger excess kurtosis. This result corresponds to the well known fact that large price fluctuations are more likely to occur in less active markets, characterized by a smaller order flow. It is also consistent with results from various market microstructure models where a larger order flow enables easier regulation of supply and demand by the market maker. It is interesting that we find the same qualitative feature here although we have not explicitly integrated a market maker in our model. This result should be compared to the observation in [17] that, even after accounting for heteroskedasticity, the conditional distribution of stock returns for small firms is higher than that of large firms. Small firm stocks being characterized by a smaller order flow N_{order}, this observation is compatible with our results.

More importantly, Eq. (15) shows that the kurtosis can be very large *even if the number of orders is itself large*, provided c is close to 1. Since $A(1)$ is close to $1/2$, one finds that even for $c = 0.9$ and $N_{order} = 1000$, the kurtosis κ is still of order 10, as observed on very active markets on time intervals of tens of minutes. Actually, one can show that provided $2aN$ is not too large, the asymptotic behaviour of $P(\Delta x)$ is still of form given by Eq. (6). This model thus leads naturally to the value of $\mu = 3/2$, close to the value observed on real markets. Of course, the value of c could itself be time dependent. For example, herding tendency tends to be stronger during periods of uncertainty, leading to an increase in the kurtosis. When c reaches one, a finite fraction of the market shares simultaneously the same opinion and this leads to a crash. An interesting extension of the model would be one in which the time evolution of the market structure is explicitly modeled, and the possible feedback effect of the price moves on the behavior of market participants.

5 Discussion

We have exhibited a model of a stock market which, albeit its simplicity, gives rise to a non-trivial probability distribution for aggregate excess demand and stock price variations, similar to empirical distributions of asset returns. Our model illustrates the fact that while a naive market model in which agents do not interact with each other would tend to give rise to normally distributed aggregate fluctuations, taking into account interaction between market participants through a rudimentary 'herding' mechanism gives a result which is quantitatively comparable to empirical findings on the distribution of stock market returns.

One of the interesting results of our model is that it predicts a relation between the fatness of the tails of asset returns as measured by their excess kurtosis and the degree of herding among market participants as measured by the parameter c. This relation is given by Eq.(15) .

Although we implicitly assumed that t represents chronological time, one could formulate the model by considering t as "market time", leading to a subordinated process in real time as in [13], with the difference that the underlying process will not be a Gaussian random walk.

Our model raises several interesting questions. As remarked above, the value of c is specified as being less than, and close to 1. "Fine-tuning" a parameter to a certain value may seem arbitrary unless one can justify such an assumption. An interesting extension of the model would be one in which the time evolution of the market structure is explicitly modeled in such a way that the parameter c remains in the critical region (close to 1).

One approach to this problem is via the concept of "self-organized criticality", introduced by Bak *et al* [2]: certain dynamical systems generically evolve to a state where the parameters converge to the critical values leading to scaling laws and heavy-tailed distributions for the quantities modeled. This state is reached asymptotically and is an attractor for the dynamics of the system. Bak, Chen, Scheinkman and Woodford [1] present a simple model of an economic system presenting self-organized criticality.

Note however that, for the above results to hold, one does not need to adjust c to a critical value : it is sufficient for c to be within a certain range of values. As noted above, when c approaches 1 the clusters become larger and larger and a giant coalition appears when $c \geq 1$. In our model the activation of such a cluster would correspond to a market crash (or boom). In order to be realistic, the dynamics of c should be such that the crash (or boom) is *not* a stable state and the giant cluster disaggregates shortly after it is formed: after a short period of panic, the market resumes normal activity. In mathematical terms, one should specify the dynamics of $c(t)$ such that the value $c = 1$ is 'repulsive'. This can be achieved by introducing a feedback effect of prices on the behavior of market participants: a nonlinear coupling between can lead to a control mechanism maintaining c in the critical region.

Yet another interesting dynamical specification compatible with our model is obtained by considering agents with "threshold response". Threshold models have been previously considered as possible origins for collective phenomena in economic systems [25]. One can introduce heterogeneity by allowing the individual threshold θ_i to be random variables: for example one may assume the θ_is to be *iid* with a standard deviation $\sigma(\theta)$. A simple way to introduce interactions among agents is through an aggregate variable: each agent observes the aggregate excess demand $D(t)$ given by Eq.(2) or eventually $D(t) + E(t)$, where E is an exogeneous variable. Agents then evolve as follows: at each time step, an agents changes its market position $\phi(t)$ ("flips" from long to short or vice versa) if the observed signal $D(t)$ crosses his/her threshold θ_i. Aggregate fluctuations can then occur through cascades or "avalanches" corresponding to the flipping of market positions of groups of agents. This model has been studied in the context of physical systems by Sethna *et al* [46] who have shown that for a fairly wide range of values of $\sigma(\theta)$ one observes aggregate fluctuations whose distribution has power-law behavior with exponential tails, as in Eq.(6).

These issues will be adressed in a forthcoming work.

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- **Appendices:**

Unless specified otherwise, $f(N,c) \sim g(N,c)$ means

$$\frac{f(N,c)}{g(N,c)} = 1 + \underset{N\to\infty}{o}(1)$$

uniformly in c on all compact subsets of]0,1[.

- **Appendix 1: some results from random graph theory**

In this appendix we will review some results on asymptotic properties of large random graphs. Proofs for most of the results may be found in [19] or [8].

Consider N labeled points $V_1, V_2, ...V_N$, called *vertices*. A link (or edge) is defined as an unordered pair $\{$i,j $\}$. A graph is defined by a set V of vertices and a set E of edges. Any two vertices may either be linked by one edge or not be linked at all. In the language of graph theory, we will consider non-oriented graphs without parallel edges. We shall always denote the number of vertices by N. A *path* is defined as a finite sequence of links such that every two consecutive edges and only these have a common vertex. Vertices along a path may be labeled in two ways, thus enabling to define the extremities of the path. A graph is said to be connected if any two vertices V_i, V_j are linked by a path i.e. there exists a path with V_i and V_j as extremities. A *cycle* (or loop) is defined as a path such that the extremities coincide. A graph is called a *tree* if it is connected and if none of its subgraphs is a cycle. A graph is called acyclic if all its subgraphs are trees.

Consider now a graph built by choosing, for each pair of vertices V_i, V_j whether to link them or not through a random process, the probability for selecting any given edge being $p > 0$, the decisions for different edges being independent. A graph obtained by such a procedure is termed a *random graph* of type $\mathcal{G}(N, p)$ in the notations of [8][4].

In the following, we will be specifically interested in the case $p = c/N$. Various graph-theoretical parameters of such graphs are random variables whose distributions only depends on N and c. We shall be particularly interested in the properties of large random graphs of this type i.e. $\mathcal{G}(N, c/N)$ in the limit $N \to \infty$.

The following results have been shown by Erdös and Renyi [19] and Bollobas [8]: If $c < 1$ then in the limit $N \to \infty$ all point of the random graphs belong to trees except for a finite number U of vertices which belong to unicyclic components. Moreover, the probability of a vertex belonging to a cyclic component tends to zero as $N^{-1/3}$. For describing the structure of large random graphs for $c < 1$ it is therefore sufficient to account for vertices belonging to trees; cyclic components do not essentially modify the results, except when $c = 1$.

[4] This definition corresponds to random graphs of type $\Gamma^{**}_{n,N}$ in [19] (see [19],p. 20).

More precisely ([8], Theorem V.22)

$$\overline{U} \sim \frac{1}{2}\sum_{k=3}^{\infty}(ce^{-c})^k \sum_{j=0}^{k-3}\frac{k^j}{j!}$$
$$\sigma^2(U) \sim \frac{1}{2}\sum_{k=3}^{\infty}k(ce^{-c})^k \sum_{j=0}^{k-3}\frac{k^j}{j!}$$

The above expressions are valid for $c \neq 1$.

•Appendix 2: Distribution of cluster sizes in a large random graph

Let $p_1(s)$ be the probability for a given vertex to belong to a cluster of size s in the $N \to \infty$ limit. The moment generating function Φ_1 of the p_1 is defined by :

$$\Phi_1(z) = \sum_{s=1}^{\infty} e^{sz} p_1(s)$$

We shall now proceed to derive a functional equation verified by Φ_1 in the large N limit when the effect of loops (cycles) are neglected.

Let $p_N^1(s)$ be the corresponding probability in a random graph with N vertices. Adding a new vertex to the graph will modify the pattern of links, the probability of k new links from the new vertex to the old ones being $(c/N)^k(1-c/N)^{N-k}\begin{pmatrix} N \\ k \end{pmatrix}$. As shown above (Appendix 1) the probability of creating a cycle tends to zero for large N. The constraint that no new cycles are created by the new links imposes that the k links are made to vertices in k different clusters of sizes. If $s_1, s_2, ...s_k$ are the sizes of these clusters, the new links will create a new cluster of size $s_1+s_2+...+s_k+1$.

$$p_{N+1}^1(s) = \sum_{k=1}^{N}\sum_{s_1,...,s_k=1}^{N}\begin{pmatrix} N \\ k \end{pmatrix}(\frac{c}{N})^k(1-\frac{c}{N})^{N-k}\delta(s_1+s_2+...+s_k+1-s)p_N^1(s_1)p_N^1(s_2)...p_N^1(s_k)$$

Multiplying both sides by e^{sz} and summing over s gives:

$$\Phi_1(z, N+1) = e^z[1+\frac{c}{N}+\Phi_1(z,N)\frac{c}{N}]^N$$

which gives in the large N limit:

$$\Phi_1(z) = e^{z+c(\Phi_1(z)-1)}$$

from which various moments and cumulants may be calculated recursively. The distribution of clusters sizes $p(s)$ is then given by

$$p(s) = A(c)\frac{p_1(s)}{s}$$

where $A(c)$ is a normalizing constant defined such that $\int p(s)ds = 1$.

• Appendix 3: Number of clusters in a large random graph

Let $n_c(N)$ be the number of clusters (connected components) in a random graph of size N defined as above. n_c is a random variable whose characteristics depend on N and the parameter c. In this section we will show that n_c has an asymptotic normal distribution when $N \to \infty$ and that for large N, the j-th cumulant C_j of of n_c is given by:

$$C_j \underset{N\to\infty}{\sim} \frac{(-1)^j Nc}{2}$$

From a well known generalization of Euler's theorem in graph theory

$$l(N) - N + n_c(N) = \chi(N)$$

where $\chi(N)$ is the number of independent cycles and $l(N)$ the number of links. This implies in turn

$$\overline{n_c(N)} = N(1-\frac{c}{2}) + O(1)$$

We shall retrieve this result below, and proceed to calculate higher moments via an approximation. Define the moment generating function for the variable $n_c(N)$ to be

$$\Phi_N(z,c) = \overline{e^{n_c z}} = \sum_{k=1}^{N} P_{N,c}(n_c = k)e^{kz}$$

The j-th moment of n_c is then given by:

$$\overline{n_c{}^j} = \frac{\partial^j \phi_N}{\partial z^j}(0,c)$$

Let us also consider the cumulant generating function Ψ defined by $\Phi(z) = \exp\Psi(z)$. The *j-th cumulant* of the distribution of n_c may then be calculated as

$$C_j(N,c) = \frac{\partial^j \Psi_N}{\partial z^j}(0,c)$$

We will now establish an approximate recursion relation between Φ_N and Φ_{N+1}. Take a random graph of size N, the probability of a link between any two vertices being $p = c/N$. In order to obtain a graph with $N+1$ vertices, add a new vertex and choose randomly the links between the new vertex and the others. Note that since in a graph of size N the link probability is $p_N = c/N$, our new graph will correspond to a graph of size $N+1$ with parameter $c' = c(N+1)/N$ so that the link probability is $c'/(N+1) = c/N$. We will assume that the probability of two links being made to the same cluster is negligible i.e. that no cycles are created by the new links, which is a reasonable approximation given the results in Appendix 1. In this case, each k links emanating from the new vertex will diminish the number of clusters by $k-1$, giving the following recursion relation:

$$P_{N+1,c'}(n) = (1-\frac{c}{N})^N P_{N+1,c'}(n+1) + \sum_{k=1}^{N} \left(\begin{array}{c} N \\ k \end{array} \right) (\frac{c}{N})^k (1-\frac{c}{N})^{N-k} P_{N+1,c'}(n+k-1)$$

Multiplying each side by e^{nz} and summing over $n = 1..N$ gives:

$$\Phi_{N+1}(z,c') = e^z \Phi_N(z,c)[1 + \frac{c}{N}(e^{-z}-1)]^N$$

or, in terms of the cumulant-generating function Ψ_N:

$$\Psi_{N+1}(z, c(1+\frac{1}{N})) = z + \Psi_N(z,c) + \ln(1 + \frac{c}{N}(e^{-z}-1)]^N \qquad (*)$$

When $N \to \infty$, an first order expansion in $\frac{1}{N}$ gives:

$$\Phi_{N+1}(z,c) + \frac{c}{N}\partial_2 \Phi_{N+1}(z,c) = e^z \Phi_N(z,c) \exp(c(e^{-z}-1)[1 - \frac{c^2(e^{-z}-1)^2}{N}] + o(\frac{1}{N}) \qquad (**)$$

where ∂_2 denotes a partial derivative with respect to the second variable. The second term on the left hand side stems from the expansion in the variable

$c' = c(1 + 1/N)$ and reflects the fact that the probability for a link has to be renormalized when going from a N-graph to a N+1 - graph.

By taking successive partial derivatives of (*) and (**) with respect to z one can then derive recursion relation for the moments and cumulants of n_c. Let us first retrieve the result given in appendix one for $\overline{n_c}$. Define $\gamma(c)$ such that

$$\overline{n_c} = \frac{\partial \phi_N}{\partial z}(0, c) = \gamma_1(c)N + O(1)$$

Substituting in (*) yields a simple differential equation for γ_1:

$$\gamma_1(c) + c{\gamma_1}'(c) = 1 - c$$

whose solution is $\gamma_1(c) = (1 - c/2)$ i.e.

$$\overline{n_c} = (1 - \frac{c}{2})N + O(1) \qquad \frac{\overline{n_c}}{N} \rightarrow 1 - \frac{c}{2}$$

Let us now derive a similar relation for the variance $\sigma^2(N, c) = var(n_c)$. Let

$$\sigma^2(N, c) = \gamma_2(c)N + O(1)$$

By taking derivatives twice with respect to z in (*) and setting $z = 0$ one obtains, up to first order in $1/N$:

$$\gamma_2(c) + c{\gamma_2}'(c) = c$$

whence $\gamma_2(c) = c/2$ By calculating the j-th derivative in (*) with respect to z one can derive in the same way an asymptotic expression for the j-th cumulant of n_c:

$$C_j \underset{N \rightarrow \infty}{\sim} \frac{(-1)^j Nc}{2}$$

Note that the asymptotic forms for cumulants of n_c are identical to those of a random variable Z with the following distribution:

$$P(Z = k) = \frac{(\frac{Nc}{2})^{N-k}}{(N-k)!} e^{-\frac{Nc}{2}}$$

i.e. $N - Z$ is a Poisson variable with parameter $Nc/2$. Without rescaling, this distribution becomes degenerate in the large N limit. Nevertheless for finite N both Φ_N and Ψ_N are analytic functions of z in a neighborhood of zero. Consider now the rescaled variable:

$$Y_N = \frac{n_c - N(1-\frac{c}{2})}{\sqrt{Nc/2}}$$

Y_N has zero mean and unit variance and its higher cumulants tend to zero:

$$\forall j \geq 3, C_j(Y_N) \underset{N\to\infty}{\to} 0$$

The standard normal distribution is the only distribution with zero mean, unit variance and zero higher cumulants. Under these conditions, one can show [22] that the convergence of the cumulants implies convergence in distribution:

$$\frac{n_c - N(1-\frac{c}{2})}{\sqrt{Nc/2}} \underset{N\to\infty}{\sim} \mathcal{N}(0,1)$$

- **Appendix 4: distribution of aggregate excess demand**

In this appendix we derive an equation for the generating function ofthe variable Δx which represents in our model the one-period return of the asset. The relation between Δx and other variables of the model is given by equation (5):

$$\Delta x = \frac{1}{\lambda}\sum_{\alpha=1}^{n_c} W_\alpha \phi_\alpha = \frac{1}{\lambda}\sum_{\alpha=1}^{n_c} X_\alpha$$

where n_c is the number of clusters or trading group i.e. the number of connected components of the random graph in the context of our model. n_c is itself a random variable, whose cumulants are known in the $N \to \infty$ limit (see Appendix 3). As for the random variables X_α, their distribution is given by Eq.(8)

$$P(\Delta x = x) \quad = \quad \sum_{k=1}^{N} P(n_c = k) \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) (2a)^j (1-2a)^{k-j} f^{\otimes j}(\lambda x)$$

In order to calculate this sum, let us introduce the moment generating functions for Δx and X'_α:

$$\tilde{f}(z) = \sum_s f(s) e^{sz} \qquad \mathcal{F}(z) = \sum_s P(\lambda \Delta x = s) e^{sz}$$

Multiplying the right hand side of the equation above by $e^{\lambda x z}$ and summing over $s = \lambda x$ yields:

$$\begin{aligned} \mathcal{F}(z) &= \sum_{k=1}^{N} P(n_c = k)[1 - 2a + 2a\tilde{f}(z)]^k \\ &= \Phi[\ln(1 + 2a(\tilde{f}(z) - 1))] \\ &= \exp[\Psi(\ln(1 + 2a(\tilde{f}(z) - 1))] \end{aligned}$$

where $\Psi(z)$ is the cumulant generating function of the number of clusters defined in appendix 3. Ψ is an analytic function whose series expansion is given by the cumulants of n_c

$$\Psi(z) = Nz + \frac{Nc}{2} \sum_{k=1}^{\infty} \frac{(-z)^j}{j!} = Nz + \frac{Nc}{2}(e^{-z} - 1)$$

one can evaluate the above sum in the large N limit as

$$\begin{aligned} \mathcal{F}(z) &= \exp[\Psi(\ln(1 + 2a(\tilde{f}(z) - 1))] \\ &= \gamma^N \exp[\frac{Nc}{2}(\frac{1}{\gamma} - 1)] \end{aligned}$$

where

$$\gamma = [1 + 2a(\tilde{f}(z) - 1)]$$

Recall that $2a$ corresponds to the fraction of agents who are active in the market in a given period. Therefore $2aN$ is the average number of buy and sell orders sent to the market in one period. We shall choose $a(N)$ such that in the limit $N \to \infty$ the number of orders has a finite limit, which we will denote by N_{orders} : $2aN \to N_{orders}$. More precisely if we assume that $2a = N_{orders}/N + o(1/N)$ then

$$\begin{aligned} \gamma^N &= \exp[N_{orders}(\tilde{f}(z) - 1)] + O(\frac{1}{N}) \\ (\frac{1}{\gamma} - 1) &= -\frac{N_{orders}(\tilde{f}(z) - 1)}{N} + o(\frac{1}{N}) \end{aligned}$$

in the above expression gives:

$$\begin{aligned}\mathcal{F}(z) &= \gamma^N \exp[\frac{Nc}{2}(\frac{1}{\gamma}-1)] \\ &= \exp[N_{orders}(\tilde{f}(z)-1)]\exp[\frac{-CN_{orders}}{2}(\tilde{f}(z)-1)] \\ &= \exp[N_{orders}(1-\frac{c}{2})(\tilde{f}(z)-1)] + O(\frac{1}{N})\end{aligned}$$

One finally obtains:

$$\mathcal{F}(z) \simeq \exp[N_{order}(1-\frac{c}{2})(\tilde{f}(z)-1)]$$

Let us now examine the implication of the above relation for the moments of D and Δx. Expanding both sides in a Taylor series yields:

$$\begin{aligned}\mu_2(D) &= N_{order}(1-\frac{c}{2})\mu_2(X_\alpha) \\ \mu_4(D) &= N_{order}(1-\frac{c}{2})\mu_4(X_\alpha) + 3N_{order}^2(1-\frac{c}{2})^2\mu_2(X_\alpha)^2\end{aligned}$$

which implies that the kurtosis $\kappa(D)$ of the aggregate excess demand is given by

$$\kappa(D) = \frac{\mu_4(X_\alpha)}{N_{order}(1-\frac{c}{2})\mu_2(X_\alpha)}$$