Price Variations in a Stock Market with Many Agents

P. Bak[1], M. Paczuski[1], and M. Shubik[2]

[1] *Department of Physics, Brookhaven National Laboratory, Upton, NY 11973*

[2] *Cowles Foundation for Research in Economics, Yale University*

P.O. Box 208281 New Haven CT 06520

Large variations in stock prices happen with sufficient frequency to raise doubts about existing models, which all fail to account for non-Gaussian statistics. We construct simple models of a stock market, and argue that the large variations may be due to a crowd effect, where agents imitate each other's behavior. The variations over different time scales can be related to each other in a systematic way, similar to the Levy stable distribution proposed by Mandelbrot to describe real market indices. In the simplest, least realistic case, exact results for the statistics of the variations are derived by mapping onto a model of diffusing and annihilating particles, which has been solved by quantum field theory methods. When the agents imitate each other and respond to recent market volatility, different scaling behavior is obtained. In this case the statistics of price variations is consistent with empirical observations. The interplay between "rational" traders whose behavior is derived from fundamental analysis of the stock, including dividends, and "noise traders", whose behavior is governed solely by studying the market dynamics, is investigated. When the relative number of rational traders is small, "bubbles" often occur, where the market price moves outside the range justified by fundamental market analysis. When the number of rational traders is larger, the market price is generally locked within the price range they define.

I. INTRODUCTION

The literature on competition among firms and the literature on stock market prices has been for the most part treated separately. Here we adhere to this approach, although eventually a model which makes explicit the feedbacks between the "real" or physical economy of firms and production must be linked to the paper economy of finance. Even the simplest of economic models tends to become enormously complicated when attempts at realistic modelling are made. Mathematical tractability as well as clarity can be lost in a welter of detail. The basic idea of our investigation is to select model segments of the estimated minimal structure needed to make an adequate description of the statistical properties of price variations in a stock market.

A motivation for this work is suggested by the early observations of Mandelbrot (1963, 1966,1967) on the nature of stock prices and the recent behavior of the derivatives markets where "10 sigma events" have been happening with sufficient frequency to raise doubts about existing models, which all fail to account for non-Gaussian variations in price. Mandelbrot observed that price variations of many market indices over different, but relatively short time intervals could be described by a stable Levy distribution, rather than being Gaussian. The Levy distribution has substantially more weight for large events than the Gaussian distribution where large events (beyond $\sim 4\sigma$) are prohibitively unlikely.

Mandelbrot's observation has since been augmented by fitting to a "truncated Levy distribution" where the scaling regime for price fluctuations is actually finite rather than infinite. Empirical observations suggest that the succession of daily, weekly, and monthly distributions progressively converge to a Gaussian (Akgiray and Booth, 1988). The presence of an intermediate scaling regime, where the price changes Δp occur with a probability distribution $P(\Delta p) \sim (\Delta p)^{-\alpha-1}$, can mathematically account for slow convergence to a Gaussian distribution at long time scales. For example, the data for the S & P 500 index is reasonably fitted by a truncated Levy distribution with $\alpha \simeq 1.4$ over a time scale which ranges from a minute to a day, with convergence to a Gaussian at approximately one month

(Mantegna and Stanley, 1995). Also, Arneodo et al (1996) observed a $1/f^2$ power spectrum at long time scales consistent with Gaussian behavior; while at short time scales truncated Levy behavior was observed. They analyzed the DEM-USD exchange rate from October 1991 - November 1994. Although this Levy-type behavior is by now well documented empirically, there is as yet no mathematical model of a stock market which can explain the origin of large price variations with "fat tails", at least over a fairly broad range of time scales. Here such a model is introduced.

We construct an extremely simple, but completely defined, economic model of many agents trading stock. These agents form a market that exhibits large variations in prices resulting from differences in the agent's behavior. The agents are of two types: type (1) are "noise" traders whose current volatility may depend on recent changes in the market and whose choice of price to buy or sell may imitate choices of others; type (2) are "rational" agents each optimizing their own utility functions. There is only one type of stock and each agent can own at most one share. These simplifications are quite drastic for many reasons. First of all, in reality the price changes of different stocks are correlated to each other. Thus one cannot treat each stock as an independent market. In addition, we are ignoring price variations due to exogenous changes such as in interest rates, money supply, or wars breaking out. Finally, we have an extremely simple description of the individual agent's behavior.

In spite of these gross simplifications, some of our toy models exhibit a statistical pattern of price variations consistent with that observed empirically. This suggests that we may have captured a dynamical process that is sufficiently robust to describe price variations in real markets. One version of the model exhibits power law fluctuations at small time scales leading to a "Hurst" exponent $H \simeq 0.6$. Eventually at long time scales the fluctuations may converge to Gaussian with $H = 1/2$. The fat tails in the probability distribution for price variations in our model are a collective effect resulting from many different interacting agents. Our results suggest that large fluctuations in price may be endogenous to the dynamics of stock markets.

A. Review of Previous Literature

Zipf (1949) wrote a provocative book in which he observed regularity in the distribution of most frequently used words in English and several other languages. He also suggested that the size of an item of rank I, by some measure was given by $I^{-\alpha}$ where α is an empirically determined exponent often near 1. Zipf applied his data analysis to diverse areas of human activity including the sizes of cities, corporations, income of individuals, etc. More recently Mandelbrot (1963, 1966, 1967) in several articles noted that the distribution of certain financial market prices appeared to be best represented by a Pareto-Levy distribution. In the tail of the distribution the probability of price change of any size was present as a power law. A related observation was that the temporal correlations in price fluctuations were characterized by a Hurst exponent $H \simeq 0.6$ larger than that for a random walk ($H = 1/2$). Recently, the truncated Levy distribution has been used to describe various economic indices, as described above. These observations have thus far confounded all attempts at a general explanation in terms of a model economy or market.

Ijiri and Simon (1977) in a work covering articles written from 1955 to 1975 considered the size distribution of firms where the driving mechanism is constant returns at any size, as suggested by Gibrat. More recently Arthur (1989, 1990) looking at production has considered models of market share based on what might be called "network increasing returns to scale." He considered the probability of future market share to be dependent on the relative sizes of current market share and offered a Polya process for updating the change in probabilities of future market share. Positive feedback leads to "history dependence" where the dynamics gives many possible outcomes for market share rather than a unique noncooperative equilibrium solution. In yet another different but related approach Bak and Paczuski (1995) discuss the role of power laws in many sciences, including economics, noting the tendency of large dynamical systems to organize themselves into a critical state with avalanches or punctuation of all sizes. Bak, Chen, Scheinkman and Woodford (1993) have provided an elementary production and inventory model which exhibits self-organized criticality (Bak, Tang, and Wiesenfeld, 1987,1988). Based on an analogy with traffic flow, Paczuski and Nagel (1996) have speculated that a critical state with fluctuations of all sizes may be the most efficient state that can be achieved in an economy.

It is with this body of literature as a basis that we attempt to set up an elementary process model of interacting agents forming a stock market. Ideally it would be desirable to construct a completely closed model with the feedbacks between the producers and consumers fully specified, but we suggest below that, as a first approximation, the behavior of stock prices can be examined without going fully to this level of complexity.

B. Outline

In Section II, the model is defined together with a description of the agents and their strategies. Section III notes the false dichotomy between game theoretic and behavioral solutions which is often made. We stress that there is no simple comfortable "right way" to construct an expectations updating algorithm. The standard noncooperative equilibrium analysis finesses this problem of dynamics.

In Section IV, we present the results and an analysis of a series of simulations of stock market trading with "noise traders" who may imitate others as well as "rational" optimizing agents who make up their own mind. When the noise traders do not imitate, but act in a purely random manner, we find that the model corresponds to the universality class of the reaction diffusion process $A + B \to 0$. Barkema, Howard, and Cardy (1996) have shown analytically that scaling exists for the temporal variations in the position of the A-B interface, which corresponds to the price in our model, with a nontrivial Hurst exponent $H = 1/4$ that is subdiffusive. When the noise agents can imitate others, we escape this universality class and find different behavior that is more characteristic of real markets. In this case, we obtain a broad distribution of price changes with an effective Hurst exponent $H \simeq 0.6$ at short times, consistent with a Levy or truncated Levy distribution, converging to a Gaussian with Hurst exponent $H = 0.5$. A particular interesting case is one where the volatility of the market price is fed back to the agents. An apparent scaling regime exists with an effective Hurst exponent $H \simeq 0.65$. The distribution of price changes is far from Gaussian, with a fat tail of price variations far exceeding the Gaussian values, very similar to data for real stock markets. We also address issues as to whether the rational traders can lock the price in the case where noise traders and fundamental analysts coexist, or if below a certain concentration a "depinning" transition occurs with price bubbles away from the rational expectations range provided by the optimizing agents.

Section V is devoted to a discussion of many of the open questions and several promising approaches to the study of stock market price variations. Section VI contains concluding remarks of our approach and potential extensions.

II. DEFINITION OF THE MODEL

Models in finance and in monopolistic or oligopolistic competition tend to be open models which are implicitly embedded in a large outside economy (see for example Sharpe (1985)). They are presented without any particular need for either feedback from the outside or conservation conditions on money to be satisfied. In finance the inputs from the firms behind the stock market are represented by lottery tickets and the only commodity traded is money. The justification for this approach is that this simplification is sufficient to capture enough detail to develop an adequate theory of investment behavior. This ignores the role of fundamental securities analysis. However, we follow this approach in order to examine for the possibility that even at this level of simplicity some insight can be gleaned concerning the nature of the distribution of price changes.

Prior to commencing any mathematical analysis, a justification for the modelling choices made is provided. In an actual economy there are many different locations or sources of uncertainty. In our models we attempt to isolate several of these sources and to examine their influence on fluctuations in stock prices. Key to our approach is to consider the dynamics of mass markets with diverse agents who may imitate.

Heuristically we offer the following sketch of the extremely simple economy we model.

A. The Firm and Stock

In our model, the presence of a single firm is only implicit in a lottery ticket or shares of the firm. The ownership of a share gives the individual a dividend at time intervals Δt. The size of the dividend is a random variable, for instance, determined by a Bernouilli process which pays A with probability ρ and B with probability $(1-\rho)$. For simplicity we consider that each individual may buy or sell one share. The share is traded *ex dividend.*

The dividend distribution affects the dynamics we are interested in only through the utility function to be defined below. However, if we wish to do the bookkeeping for the profits and losses of the traders, we assume that at each instant every agent who owns stock is paid the average dividend. This regularization does not affect our evaluation of the agents' performance in the long run where dividend is paid many times. Of course, the liquidity of the agents is affected by the random variations of dividend payments, but we assume that this does not affect the agents' behavior.

B. The Agents

In the models below we consider two types of agents. Both types are gross simplifications but they provide enough structure to carry out a reasonably complex investigation. The first type may be characterized as noise traders who learn essentially nothing about equilibrium economics. Each of these traders starts with an estimate or "guesstimate" of a price at which he is willing to sell or a price at which he is willing to buy a unit of stock. At each time step, a trader is chosen and will update his bid or offer by one unit, approaching the current price with probability $(1+D)/2$ and moving away from the current price with probability $(1-D)/2$. Eventually an overlap occurs between a chosen buyer (seller) and any other seller (buyer) and a transaction takes place.

The current seller (buyer) now becomes a buyer (seller) at a new price. Equivalently, one may think of this as a new buyer and seller entering the market, with the old ones that traded removed. The new price is either chosen randomly within a finite interval, or the new buyer (seller) picks another buyer (seller) at random and copies the same price to buy (sell). In general this type of "urn" or copying process is responsible for positive feedback and path dependence as noted by Arthur (1990). The urn process also can create a peaked structure of organized agents leading to avalanches of all sizes as found in a different context by Manrubia and Paczuski (1996). Here we find that the positive feedback inherent in the urn mechanism we are proposing gives rise to anomalously large fluctuations that grow in time faster than a random walk over a broad range of time scales, before eventually converging to the Gaussian result.

Each type 1 individual is a simplistic trader who might lose money if he churns too much at poor prices; but if the dividend rate is sufficient, relative to the price he pays he could make money. The presence of dividends converts the process into a nonzero sum game and all could profit if they did not overpay for a stock. We assume that there are $N-K$ traders of this type and K traders of the next type. There are $N/2$ shares available; hence half of the population are potential buyers and half potential sellers. The initial endowments of an individual j of type 1 are $(1, M(j,0))$ if he owns stock, and $(0, M(j,0))$ if he owns no stock. Each individual who owns a share is a potential seller and each individual who does not own a share is a potential buyer. It is not allowed to own more than one share. As we are not postulating an explicitly optimizing behavior for these agents we do not need to specify the (in general, unobservable) utility function for each agent. Although if we wish to "keep score" we may do so.

1. An Optimizing Agent with a Utility Function

We assume that there is another type of trader who maximizes his short run or period by period utility function which might be of the form

$$U = \nu \min[A, B] + (1-\nu)(\rho A + (1-\rho)B) \quad . \tag{1}$$

The interpretation of this equation is that each individual has an overall risk averse utility function which is a mix of the extreme of risk aversion ($\nu = 1$) and risk neutral ($\nu = 0$). Indexing ν as $\nu(j)$ for the j'th individual, we may give each individual a different risk profile. The traders buy or sell one unit of stock or do not trade. They invest any extra cash in a savings account which pays a fixed rate of interest i per period.

2. Initial Conditions

We assume that there is a fixed given rate of interest on deposits $= i$. The $n(n = |N|)$ traders each start with a different price expectation distributed in a rectangular manner over the interval $[p_-, p^+]$. $(N-K)/2$ traders of the first type own a single share and $K/2$ of the second type own a single share.

3. Strategy

In this highly simplified model no bank or other loan market is assumed to exist. We assume implicitly that all agents have more than enough money to buy a share and that any money not spent in the market is swept into an interest bearing account. In a more complex and more realistic model it is natural to introduce a commercial and central banking system designed to accept deposits or lend at a fixed rate of interest i. This permits the system to create money in a model with an outside bank.

The first type of agent has a decision rule which can hardly be called a conscious strategy, but may have a tendency to follow the decisions of others in the market. Thus when a buyer who is an imitator purchases a share and now becomes a seller, his initial offering price is chosen by picking the price of a random seller.

The second type of agent buys or sells stock if he thinks its dividend return is high enough above or below the interest rate. This can be expressed as:

$$\text{Buy if} \quad \frac{U}{p_s} > i + \Delta_1 \quad ; \tag{2}$$

$$\text{Sell if} \quad \frac{U}{p_b} < i + \Delta_2 \quad . \tag{3}$$

The parameters Δ reflect decision "stickiness". The distribution of utility functions for the agents may be derived from a distribution of (ν) and (ρ) for the agents. This leads to a distribution of buying and selling prices for the heterogeneous agents. An agent's strategy is to name two prices p_s at which (or better) he will sell and p_b at which level (or cheaper) he will buy. These strategies can be regarded as having been derived from an optimization on the one period utility functions or otherwise considered merely as another form of stock market behavior.

III. GAME THEORETIC OR BEHAVIORAL SOLUTION?

A simple false dichotomy between game theoretic multiperson optimization approaches and behavioral solutions can be made. This coincides with forgetting that the noncooperative equilibrium solution itself is a behavioral solution. When the individual optimization is considered, it is contingent on expectations of market price. A specification of how expectations are handled cannot be avoided. Conditioning the individual's optimization on the proposition that all individuals must have mutually consistent expectations implies the acceptance of a weak form of local optimization defined only at a fixed point in the system.

If one considers only equilibrium with consistent expectations it is possible to engage in a complete finesse of all dynamics. The magic invocation of rational expectations is nothing more than another way of stating that a noncooperative equilibrium exists. It sidesteps the problem of the formation of expectations by the argument that given the appropriate expectations concerning prices the expectations will be self validated by optimizing behavior based on their acceptance. Nothing is said about convergence to equilibrium if the system is not already there. No learning or other behavioral theory is supplied to indicate how expectations are formed out of equilibrium.

In essence, a game theoretic solution to an n-person game in extensive form is nothing more than a set of n strategies which complete the description of the motion of the system. Any logically consistent set will do, thus any well-defined behavioral model will suffice. The justification or rationalization as to why one wishes to consider one solution better than another poses substantive questions about what we wish to consider as rational economic behavior.

A. Expectations

As a game is played forward sequentially one cannot avoid dealing with the formulation of expectations. If one's only concern were with equilibrium we can invoke a consistency condition requiring that if an equilibrium exists all individuals' expectations must be consistent so that no one is motivated to change strategy. This may be interpreted as the rational expectations assumption, or alternately as postulating a noncooperative equilibrium. However neither assumption provides information on the nature of dynamics. Even given this condition, there is no guarantee that an equilibrium will be unique. In the approach here we consider both the game theoretic solution of noncooperative equilibrium for the dividend optimizers and a "mass particle behavioral approach" of the others.

B. Information Conditions and Expertise

A model of this variety may be highly sensitive to information conditions. The size of strategy sets increases astronomically with knowledge of contingencies. By letting an individual obtain information about some of the randomizations he may be able to foresee the future dividend of the firm. If information differentials are introduced then the number of different types of players is increased as they are now differentiated by information.

C. Loans and Bankruptcy

If it were possible for the individuals to borrow each period then we would have to take into account the possibility that they could go bankrupt each period after the first. In these highly simplified models we omit this feature which could increase market instability. The question of bankruptcy is currently being studied by Geanakoplos, Karatzas, Shubik, and Sudderth. Even in a simple fully defined process model with strategic choice the conditions required to keep track of money and credit are somewhat elaborate and tedious. At first glance if there were a rule adding a heavy penalty to the utility function it might appear that, for a stiff enough penalty, all agents would strategically avoid default, but with uncertainty present, as long as the penalty is finite, it is easy to construct situations where there is a nonzero possibility that individuals will borrow and default occurs. When this happens rules are needed to describe how the debt is extinguished. Fortunately in these simple models these difficulties are avoided by having no borrowing and giving all plenty of money.

D. Equilibrium and Behavioral Finance

The noncooperative equilibrium and rational expectations studies are devoted to the study of equilibrium positions. By implicitly requiring that expected prices and realized prices are always consistent they throw away learning, error and all considerations of dynamics. Even if one's only interest were to examine equilibrium states of the system, a satisfactory economic model calls for a full specification of a process model. But with such a specification the stage is set to consider adjustments from disequilibrium as well as timeless equilibrium. When markets are incomplete and there is exogenous uncertainty present, at best we can only seek equilibrium conditions in the aggregate, as the trajectory of any individual will be random even if the assembly of agents shows some regular behavior. The thrust of the work here is to select a set of models complex enough to examine dynamic behavior of many competing agents yet, in some instances simple enough that we can examine the noncooperative equilibria and contrast them with the behavioral models.

E. A Few Crude Facts

In order to sweeten our intuition for this type of model, a few statistics selected from the New York Stock Exchange (NYSE) 1994 Fact Book supply some orders of magnitude. In 1994 approximately 51,000,000 individuals in the United States owned stock, and around 2,600 issues were traded on the NYSE. The firms had 140,000,000,000 shares valued at $4.45 trillion, giving an average value of around $31 per share and an average issue of around 53,000,000 shares per firm. There were around 10,000 institutional investors or intermediaries and around 120 initial public offerings raising $22,000,000,000 or approximately 1/2% of the total market value. The velocity, or turnover ratio was 54%.

F. A Trading Volume Puzzle

We observe that in the actual New York market the volume of trade has been of the order of more than 50% yet, as is noted in the example in Section IV.A, if we postulate that there is a single type of trader with indefinite life, the noncooperative or rational expectations equilibrium will involve all individuals holding an optimal portfolio of stock and never trading. What are the factors that account for stock market trading and how large are they? We consider five factors. They are: 1. overlapping generations (OLG); 2. life cycle considerations; 3. stationary risk trading; 4. the influence of exogenous events and 5. heterogeneous investment activity with nonconsistent expectations. If we consider that the period of time for which an individual owns a portfolio of stock is somewhere between 20-50 years, then the pure OLG contribution to stock trading for a constant population is somewhere between 2-5% on the conservative assumption that stock will be sold rather than kept from generation to generation. Even a casual glance at life cycle changes in expenditure such as selling stock to put children though college or selling off stock to compensate for loss of earnings in retirement probably does not account for more than 10-20%.

Karatzas, Shubik, and Sudderth (1995) and Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) utilizing somewhat different models were able to show the existence of stationary strategies for an economy modelled as a set of parallel dynamic programs. Thus there is a stationary wealth distribution for all agent which maps onto itself. The same agent in such a stationary state could be poor and risk averse at one point in time and rich and possibly risk neutral at another point in time. It is possible that agents changing wealth status in a stationary state might wish to exchange portfolios. Even if we could establish this as feasible, an open question remains as to what proportion of share trading could be attributed to this type of trade. Thus there is a basic question concerning how much of stock trading can be explained by equilibrium analysis and how much requires explanation from the dynamics of nonstationary trade.

Economic systems are subject to exogenous shock such as wars breaking out, presidents being shot, natural disasters occurring, or innovations creating new markets and wiping out old ones. All of these incidents, in one form or another, could impact on different individuals with different levels of intensity and lead to considerable arbitrage before the system settles down (if it ever does). It is not known how much trade is caused by these phenomena. The fourth and fifth points are connected. Even if we believe fully in Bayesian updating, no matter how complicated and in rational expectations or mutually consistent expectations with noncooperative equilibria, all of this apparatus does not tell us where our *a priori* expectations of when the next earthquake will occur or who will be the presidential candidate in eight years time and how these events will impact the stock market. The way the phrase "rational behavior" is used in much of economic theorizing is as a euphemism for one person optimization in a context given scenario. But much of economic activity is devoted to getting the context right. Thus the key question not asked in the rational expectations or noncooperative equilibrium analysis is how do individuals obtain a cognitive map of their problem, form expectations, and estimate consequences. The second question, answered by those who accept rational expectations analysis is that, given the context all individuals act as if they are maximizing their expected payoff in a one person optimization problem. The maximization assumption is essentially normative and although it might hold in circumstances where individuals are experts and dedicated money maximizers it does not necessarily provide a good empirical view of actual behavior. If there is a substantial segment of the market following some other rule its presence could influence the environment in which the "rational optimizers" act. Our investigations in Section IV consider this possibility.

IV. SIMULATIONS OF THE MARKET

Before discussing the various models which have been studied numerically, we first define the mechanics of the market in which the agents are playing. There is one type of stock, and each agent can own only one share. Therefore, each agent is either a potential buyer (if he does not own a share) or a potential seller (if he owns a share). For simplicity and without loss of generality, we assume that there are N agents and $N/2$ shares. Each share owner advertises a price, $p_s(j)$ at which he is willing to sell his share, and each potential buyer advertises a price $p_b(j)$ that he is willing to pay for a share. The prices may assume any value in the interval $0 < p < p_{max}$. The various types of agents differ in the way that these prices are chosen.

At each update step a single agent is chosen randomly to look into the market. For instance, imagine that he turns on his computer and observes all advertised prices in the market. The market is a list of advertised buying and selling prices for all agents. If the agent owns a share, and observes that there are one or more buyers who are willing to pay more than his advertised price, he sells to the buyer offering the highest price. That price defines the market price, $p(t)$, of the stock at that particular instant when the transaction occurred. The seller (buyer) in the transaction now becomes a potential buyer (seller) and chooses a new price p_b (p_s) at which he is willing to buy (sell) a share. If no transaction occurred, the agent may update his advertised price according to his particular rule, which may or may not be related to his observations of the market.

If the randomly chosen agent did not own a share, and there are sellers willing to sell at or below the price which he is willing to pay, he buys from the agent charging the lowest price, and now becomes a seller who advertises a selling price $p_s(i)$. If no transaction occurred, the agent may change his bidding price. This process is repeated *ad infinitum.* For simplicity, in the figures presented here, a time unit is chosen as the average number of updates for an agent. Thus, in t time units each agent has been active and looked into the market on average t times. Since the updating of the agents is (random) sequential there is no need for an intermediate agent clearing the market, as would be the case for a situation with many simultaneous bids.

A. A Market with Fundamental Value Buyers Only.

Let us first consider the situation where all agents are "rational", choosing their prices according to the utility function defined in Section II.B.1. Each agent j has a different stickiness $\Delta(j)$ and risk aversion $\nu(j)$. These variables are chosen from some arbitrary distribution which is bounded between zero and one. One sees from Eqs. 1-3 that this translates into different buying and selling prices, $p_b(j)$ and $p_s(j)$, for the agents with a distribution which is determined by the parameters in Eqs. 1-3. The buying and selling price for each agent are "quenched" random variables; i. e. they remain the same throughout the simulation. The details of the distributions of $p_b(j)$ and $p_s(j)$ are not important.

We assume that the distribution of buying prices for the agents is uniform in an interval which partially overlaps with a higher interval of uniformly distributed selling prices. After some time, the market comes to rest in a state where all $N/2$ agents who own stock have a selling price p_s above a certain price p^*, and all the $N/2$ agents who do not own any shares have a buying price p_b less than p_*, where $p_* < p^*$. Because of the stickiness Δ the position of the gap between buyers and sellers depends on the initial conditions. In this state, none of the agents have any reason to take further action. Buyers and sellers "phase separate" into nonoverlapping regions of price with a dead zone in between. Thus, with only one stock for sale, rational agents will not trade in equilibrium but hold their portfolios indefinitely. In all the numerical simulations to follow, we shall assume that the rational agents had already reached such an equilibrium state with no trade at the beginning of the simulation.

B. A Market with Simple "Noise Traders" Only.

At the other extreme, let us consider a market where all the agents trade according to their observations of the state of the market without concern for the fundamental values. In principle, their behavior can be defined in a number of different ways. For simplicity, we may initiate the simulation in a state where the $N/2$ agents who own stocks are willing to sell at prices which are uniformly distributed in the interval $p_{max}/2 < p_s < p_{max}$, and the bids of the $N/2$ agents who do not own stock are uniformly distributed in the interval $0 < p_b < p_{max}/2$.

There are many different ways to define the behavior of the noise traders. Perhaps the simplest behavioral model is when each agent's price fluctuates randomly, independent of the other selling or buying prices in the market. The agents interact only by buying and selling when there is an overlap in prices. We consider this simple model first, before investigating more complex behavioral models with a higher degree of interaction between agents.

1. Independent Noise Traders

At each update step, the price $p_s(j)$, if the chosen agent j is a seller, or $p_b(j)$, if agent j is a buyer, changes randomly by one unit with equal probability in either the downward or the upward direction. Most of the time this will not induce any trade, but occasionally the chosen agent will find himself at a price level where other buyers or sellers become interested, and a sale will take place at a price $p(t)$. After the sale, the new buyer will choose a new bidding price for possible future trade randomly between 0 and $p(t)$; the new seller will choose an asking price randomly between $p(t)$ and p_{max}. How does the price $p(t)$ vary with time?

It turns out that a very similar process has been studied extensively by physicists, and a mathematical solution exists for the fluctuations of $p(t)$ which are scale free. The Independent Noise Traders model is equivalent to a reaction-diffusion model where two types of particles, A particle and B particles, are injected at different ends of a tube. These particles each diffuse randomly until a particle bumps into a particle of the opposite type, whereupon they annihilate each other. When this reaction occurs new A and B particles are injected at opposite ends of the tube (Figure 1). The positions of the two types of particles correspond to the prices p_b and p_s respectively. The position where the annihilation event takes place corresponds to the market price $p(t)$. The reaction-diffusion process is called "A+B $\rightarrow$ 0", and has been extensively studied.

Barkema, Howard, and Cardy (1996) have shown mathematically that the variation $\Delta p(t)$ of the price after time t scales as

$$\Delta p(t) \sim t^{1/4}(\ln(t/t_0))^{1/2} \quad , \tag{4}$$

where the parameter t_0 could depend on the number of particles N and the size of the price range or "tube", p_{max}. Thus, at very long time scales the price variations are scaling with a Hurst exponent $H = 1/4$. This value is much less than the exponent for a random walk $H = 1/2$, where the distribution after a long time is a Gaussian with a width scaling as $t^{1/2}$. For shorter time scales the variations are larger because of the logarithmic factor in Eq. (4). Of course for the longest time scales the variations cease to increase because of the global confinement of the price range. We stress that the situation considered here is pretty academic in the context of economic theory. Nevertheless, it demonstrates rigorously that anomalous scaling behavior, like the one observed for real markets, can arise as a consequence of the interactions between very many agents in a simple market model. This type of scaling cannot occur in systems with a few degrees of freedom, or agents in our case. The formalism used to derive Eq. (4) makes use of Quantum Field Theory, and there is no simple intuitive argument for the anomalous (non-Gaussian) form.

a. Numerical Simulation Results The behavior of the Independent Noise Traders model is illustrated in Figure (2 a,b,c,d). The system includes $N = 500$ agents operating within a price range of $p_{max} = 500$. Figure 2a shows the variation of the price vs time. Figure 2b shows the histogram for the distribution of agents' prices at some instant in time. The 250 agents to the left of the gap at $p = 247$ are currently potential buyers, the 250 agents to the right of the gap are sellers. Note that the number of potential buyers or sellers is relatively small near the gap which defines the current market price. Agents diffusing into this regime will trade, or "annihilate", and be shifted to other values on the other side of the gap.

The price variations can be conveniently represented by a Hurst plot (Feder, 1989). The variation over a time interval t is characterized by the range $R(t)$, which is the maximum variation over nonoverlapping time intervals of length t, averaged over the entire time record of the simulation. For Gaussian processes $R(t)$ increases as $t^{1/2}$. On a log-log plot the curve for a Gaussian process would be a straight line with the slope equal to 1/2. Figure 2c shows $\log R$ vs $\log t$ for the simulation. From the theoretical result for the equivalent $\mathrm{A} + \mathrm{B} \to 0$ process, we predict $\ln R = 1/4 \ln t + \ln(\ln(t/t_0))$. Indeed, for large t the slope approaches 1/4 asymptotically, and for smaller t the apparent slope is larger because of the logarithmic corrections. The logarithmic factor represents the large noise or glitches at shorter time scales which overlay the slowly varying long time power law behavior, as seen in Figure 2a. Thus, for small t the effective exponent for the fluctuations in Figure 2c is much larger than 1/4, and remarkably even larger than the random walk value of 1/2. The logarithmic factor gives rise to slow convergence to the asymptotic value 1/4. Thus, if one had access only to short time fluctuations one might erroneously conclude that the exponent is much greater than 1/4. In Figure 2d, we have plotted both $(R^2(t)/t^{1/2})$ and $(R^2(t)/t^{1/2}(\ln t/t_0))$, with $t_0 = 1$, and find results consistent with Eq. (4).

The price variations in the independent noise traders model asymptotically have a power-law form discussed by Mandelbrot, although the exponent is smaller rather than larger than 1/2. However, at short time scales the model exhibits larger than Gaussian fluctuations as a result of the logarithmic term. These are both rigorous results.

One might naively have suspected that the model would exhibit pure random walk behavior for the price variations, since it is based on the diffusive behavior of the individual agents. Nothing could be further from the truth. Although the price variations of this model are not consistent with observations of reality, we find that the price variations exhibit scaling with a logarithmic factor responsible for slow convergence to the asymptotic result. There is excellent agreement between our numerical results and the previous analytic work on the reaction-diffusion system. We now consider what effects are able to alter the universality class of the model and take it outside the regime of the $A + B \rightarrow 0$ system.

2. Drift Toward the Current Price

We now allow the agents to adjust their current price towards the actual price of the last trade that occurred in the market. The agents become more realistic as time passes, but again, their behavior is unrelated to any fundamental value of the stock. At each update step the agent that is chosen moves one step towards the market price with probability $(1 + D)/2$, and one step away from the market price with probability $(1 - D)/2$. In the simulation, a drift coefficient $D = 0.05$ was chosen. Figures 3, a, b, c are equivalent with the corresponding figures for the purely diffusive case of independent noise traders. A finite drift does not change the general pattern, including the slow convergence towards the exponent $H = 1/4$ for the price variations.

The slow price variations are related to the fact that the prices are confined by fiat to a box of size p_{max}. These noise traders at least realize that prices have to be within a certain range - otherwise they would have no clue to choosing their prices between zero and infinity. After a trade, the current buyer and seller would choose a new price randomly. A more interesting situation arises if the agents, when choosing their asking prices and bids, mimic existing traders in the market. This type of urn process can lead to large scale organization in dynamical systems (Manrubia and Paczuski, 1996).

C. The Urn Model

We will consider an "urn" process, where after each transaction the new selling and buying prices for the trading agents are chosen by randomly picking a buyer and seller and copying his price. This has the effect that the new price is chosen with a probability proportional to the number of traders who currently have that price. However, no global information on the part of buyers and sellers is needed when choosing a new price. This opens up the possibility of mimicking crowd behavior, where agents follow each other, again without paying any respect to fundamental market values.

We modify the previous model of noise traders with drift where the agents, in addition to copying each other's price choices upon becoming a buyer or seller, exhibit a stochastic drift towards the current market price. This allows for the preservation of organized clusters of agents with similar prices. Once the market drifts away from its "true" value the price is not subject to any restoring force whatsoever.

1. Numerical Simulations of the Urn Model

Figure 4a shows that the price variations in the urn model with $N = 500$ agents are much more dramatic than in the previous case, and qualitatively look a lot more like the variations observed in a real stock market. Initially, the buying prices $p_b(j)$ were chosen randomly between 1900 and 2000, and the $p_s(j)$ between 2001 and 2100. The parameter $p_{max} = 4000$, although it could be made arbitrarily large. The collection of agents organize themselves into a well defined price range (Figure 4b) which is not affected by the external parameter p_{max} as long as p_{max} is sufficiently large. This is in contrast to the case of independent noise traders (see Figs. 2b, 3b) who fill up the entire interval $[0, p_{max}]$. At very long time scales, the center of mass of this self-organized distribution of market prices wanders up and down as a random walk. The exponent for the variations is $H = 1/2$ at long time scales.

Again, though, the convergence is very slow. Although we presently have no analytical results to refer to, we speculate that the slow convergence may be due to a logarithmic factor as for the previous model with $H = 1/4$. For relatively small t, the effective Hurst exponent is larger than $1/2$. Indeed, as referred to in the Introduction, it has recently been argued, based on real economic data, that in the long run price variations are random walk. The apparent Levy distributions with $H > 1/2$ are a transient phenomenon for short time scales only, but there is slow convergence to the Gaussian process. Remarkably we observe a very similar slow convergence to Gaussian behavior, with superdiffusive fluctuations at small time scales, in simulations of the urn model.

2. Volatility Feedback

Very interesting behavior was observed in the case where the information on the volatility of the market is fed back to the agents. It is known that that various market indices exhibit volatility clustering. At times where prices have recently been volatile, this volatility could influence the behavior of the noise traders, or the imitators. If the Dow-Jones index exhibits a large drop on a given day, it is likely that there will be a large variation the next day, although it could go either up or down. We attribute this to the reaction of the agents to the price variations. To mimic this effect, we simulated the urn model above with the added feature that the diffusion constant and drift for the agents' prices is proportional to the actual recent variations of market prices.

In the simulation, if the price change during the last period of 100 time units, is ΔP, an agent updating his price will increase or decrease his price randomly by an amount ΔP. The probability for increase is $(1-D)/2$ and decrease is $(1+D)/2$ as before. Thus, instead of moving one unit up or down, the agent moves ΔP units. Since ΔP depends on recent changes in the market price, this obviously is a mechanism for positive feedback in the economy where large fluctuations create space for large fluctuations in the future.

Figure 5 shows the price variations in the urn model with volatility feedback. Indeed, the variations are more dramatic than in any of the other cases studied. As shown in Figure 5c, there is an apparent plateau with an exponent $H = 0.65$ even for relatively long time scales.

In order to study the scaling behavior further, we plot the distribution of price differences dp at a fixed time interval dt for various values of time intervals. This was done for real markets by Mandelbrot (1963, 1966, 1967), Mantegna and Stanley (1995), and Arneodo et al (1996). One simply samples the index, e.g. the S& P 500, at regular intervals (say dt is one hour) and records the differences between subsequent measurements. Figure 5d shows the price difference distributions for time intervals dt ranging from 200 to 6400 time units in our simulation. Of course, the larger the time interval, the larger the variations. However, when plotted in terms of properly defined scaling variables, the picture becomes immensely simplified. Figure 5e shows the same data plotted as a function of the scaling variable variable $z = dp/(dt)^{0.65}$. In the Figure, the vertical axis is scaled so that the entire distribution is normalized to 1. Within statistical uncertainty, all the curves collapse onto a single curve. This data collapse shows that the distribution of price variations $P(dp, dt)$ exhibits scaling behavior,

$$P(dp, dt) \sim F\Big(dp/(dt)^{0.65}\Big) \quad , \tag{5}$$

i. e. it can be expressed in terms of one scaling variable instead of two independent variables. This was precisely the behavior observed by Mandelbrot, who suggested that the scaling function F is a stable Levy distribution.

Figure 5f shows the scaling function as obtained for $dt = 1600$. For comparison a Gaussian with the same variance is also shown. Note the dramatic difference. In particular, the distribution shows a fat tail indicating a significant probability of observing fluctuations which greatly exceeds the standard deviation. The data is insufficient to determine whether or not the scaling function is precisely a Levy function with power law decay. The value of the exponent in the scaling function is very similar to the one observed for real markets. The time intervals over which there is scaling spans the interval from $dt = 200$ to $dt = 6400$. The upper limit corresponds to a time interval during which each trader on average performs a few trades. The position of the center of mass of the distribution of agents, Figure 5b, is not significantly shifted during that interval. At larger timescales, the distribution shifts, and the price variations must become Gaussian. This simulation is our most convincing argument that the scaling of price variations in real markets indeed has its origin in the collective "crowd" behavior of very many agents interacting with each other by imitating and watching the same market data, irrespective of underlying fundamental values.

In the remaining numerical simulation studies we do not include the volatility feedback effect in the noise traders, but revert to the simple urn model.

D. A Market with Fundamental Value Buyers and Noise Traders with Imitating Behavior

It is natural to consider what happens in a market with both rational optimizing agents and noise traders who imitate others. In a real market situation, we might expect both types of players, as well as many other types. In our computer laboratory, we can ask some basic questions about the market. For example, will the existence of rational players be sufficient to discipline the noise traders so that they can get a free ride, or will the rational agents be able to systematically exploit these traders? Maybe the noise traders change the dynamics sufficiently, so that the rational traders have to take their behavior into account.

In the initial setup, the noise traders' prices are distributed in an interval around the median p^*, limited by p_{min} and p_{max}. Half of the rational traders own a share and their asking prices are distributed randomly in an interval above p^*. Their potential buying prices, which become relevant if they happen to sell their share is also fixed throughout the simulation. They are an amount Δ lower, with Δ distributed randomly between 1 and Δ_{max}. Similarly, the second half have buying prices in an interval below p^*, and if they happen to buy, their potential selling price will be an amount Δ higher than their buying price. The noise traders are not aware of who is a noise trader and who is rational. When they are selecting new prices after buying or selling, they randomly choose the price of any other agent, so that both the prices of rational and noise traders are copied. In addition to the actual transaction, this provides a coupling between the two types of traders.

1. Numerical Simulation Results

Figure 6 shows the results for a simulation with a small fraction, 2%, of rational traders. All the prices of the rational traders are arbitrarily confined to the interval between 1931 and 2068. This interval is supposed to represent the interval spanned by the utility function in Section II.B. The highest price can be thought of as the price with zero risk aversion, plus the stickiness Δ.

In the beginning of the simulation, the prices of the two types of agents are similar, but eventually the noise traders convince themselves that the stock is worth more than it is, and their prices escape to higher values. At that point, all the rational traders have sold their shares; they are out of the market. Eventually the market price returns to the window of the rational players. This behavior, where the crowd effect leads to unreasonably high prices is known as a "bubble". At least in our model, we observe that this is a possible outcome in a market with few rational traders. The price variations at long time scales follow a power law with exponent $H = 1/2$ just as for the case with noise traders only in the urn model.

Figures 7a,b shows the situation with 20% rational traders. Now, the price range of all traders is confined within the range of the rational traders. Only very brief deviations occur, and common sense is rapidly restored. Figure 7b shows the skew distribution of prices during the brief excursion that occurred around 4000 on the horizontal axis of Figure 7a. At that point, all of the fundamental traders own stock and most of the noise traders are undervaluing the stock.

De Long, Schleifer, Summers, and Waldmann (1990), in a stimulating analytical paper, have considered an economy with optimizing agents and noise agents. In their analysis they observed that under some circumstances the noise agents might outperform the rational optimizers. If the optimizers are risk averse, the presence of random agents adding to the variance in price will make the optimizers more cautious that they would be otherwise.

We have measured the performance of all the agents in the simulation, extending over 1 million time units. During that interval, each agent on average was updated 1 million times. The noise traders have traded on average 4100 times, while the fundamental value traders only 80 times. Thus, their activity is only 2% of that of the noise traders. Actually 5 of the 100 rational traders did not sell at all during the simulation.

There are two sources of profits, dividends and capital gains. The average capital gain for the fundamental value traders was 187, compared with a loss of 35 units for the noise traders. The rational traders made their capital gains simply by buying low and selling high, whenever possible, and doing nothing in the meantime. Some of the rational agents were lucky enough to make significantly more that the average due to a strategically favorable choice of buying and selling prices near the equilibrium around 2000. One agent made a profit of 1029 units. The noise traders share the losses in a much more democratic way, with no significant variations from agent to agent because their prices are changing all the time while the rational agents prices are quenched and fixed throughout the simulation.

In order to estimate the dividends relative to the interest rate, we assume that the highest selling price, $p = 2150$, among all the rational agents is the price for which the dividend, on average, balances the interest rate. Thus, if you buy a share for $p = 2150$ you make no profit from dividends. If you buy at a lower price p, you make a profit by keeping the difference $2150 - p$ in the bank. With an interest i paid every t time steps, the profit becomes $(2150 - p)i/t$ per time step. Assuming somewhat arbitrarily that $i/t = 1/1000000$, the average profit from dividends for the noise traders is 72 units, the average profit from the rational traders was 83 units. Thus, there appears to be no significant difference. This is due to the fact that all the trading action is rather close to the equilibrium value of 2000, so the average profit from dividends for all agents during the 1000000 time steps is 75, corresponding to half the agents holding shares and making profits at any given time.

However, this statement is deceiving. The profits from dividends of some rational agents were twice as large, obtained by simply holding on to the stock throughout the simulation. Others did less well by offering too low prices and never entering the market. The time scale for the total simulation can be thought of as the time scale for which the interest payment is equal to the stock price, of order 10 years. The time unit is 1/1000000 of that, of order 5 minutes.

E. A market with Experts Introduced

We may model certain aspects of expertise by considering that some of the agents are more adept at "picking winners" than others. A way in which aspects of expertise can be modelled is to let some individuals know the outcome of certain random variables when others are not yet informed. This, in essence, enlarges their strategy sets. If we did this here the advantage goes to the better informed. The strategy sets of the experts are larger than the other optimizing agents as objective uncertainty about next period's dividend is removed. However, their uncertainty about the behavior of the noise traders still remains. Keynes' observation about wanting to know what the average opinion of what the average opinion will be still holds. Dubey, Geanakoplos, and Shubik (1987) were able to show analytically the gain to the better informed relative to the less informed of the same type in a game with rational traders only. We do not simulate variations in expertise as this analysis suggests that qualitatively, beyond observing that the more expert of a specific type will do better than the less expert, no new phenomena will appear.

F. Survival of the Fittest?

A simplistic view of competition would be that the optimizers and experts would eventually drive out the less skilled. However, the stock market is not a zero sum game and, as in predator-prey species relationships, it is feasible for both species of traders to survive, as seen in the numerical simulations above. Even in games which are zero sum in money, there are after all individuals who buy lottery tickets or play slot machines for their entire lives, while others sell lottery tickets or own slot machines.

G. Trading with Many Stock

For simplicity we have limited our considerations thus far to trading in one security. A natural generalization of our modelling would be to consider trade with many securities. We note that a distinctively new phenomenon appears in proceeding from one to two or more stocks. The noncooperative equilibrium of the strategic market game for the optimizers only may longer be unique even with trade in only two different stocks, both of which pay out in ownership claims to the same single consumer good. Furthermore at an equilibrium point, there may be stock trading taking place, unlike the case for a single stock. We defer the investigation of trade with many stocks to future investigation.

V. MANY MODELS OF MARKET PRICING

The models presented in this section provide an opportunity only to scratch the surface of a highly complex set of problems. In this section we briefly comment on other approaches and phenomena which merit consideration.

A. Experts, Intermediaries, and Social Networks

Another potential source to consider as contributing to the dynamics of stock market prices may come through the use of intermediaries. One of the little understood phenomena in social psychology is the behavior of the crowd. The classical work of Le Bon (1982) written over a hundred years ago is still possibly the most perceptive writing on the nature of crowds. When a panic breaks out it is as though a mass of individually independent particles were all simultaneously polarized.

In the few statistics noted in Section III.E we observed that there are around 10,000 financial intermediaries trading on the NYSE. Institutions such as pension funds, mutual funds and insurance companies hold well over 50% of all equities (see NYSE Fact Book, 1994, p.83). More and more the individual investor invests through an intermediary who supplies credit, liquidity, matching, accounting, record keeping, transactions costs savings, information and, in some instances, expertise.

A natural extension of our models is to include an extra layer of intermediaries. An extreme model has all 50,000,000 individual agents as secondary sensors who spend their time trying to select the best of the 10,000 intermediaries who spend their time trying to buy the best portfolio of the 3,000 firms. The variation in expertise lies primarily in the 10,000 intermediaries. Historically, many of the intermediaries were partnerships or mutual firms, but recently many have become corporations with stock traded on the major exchanges. Banks and insurance companies appear, on the whole, to have been longer lived than stock brokerage houses, where often success depended on one expert individual. As a first crude approximation one could have a birth death process for a financial intermediary to be the same as the manufacturing firm. Left out of these models are several further basic sources of fluctuations in the economy. They are the role of credit granting (in 1994 margin debt alone was around $60 billion or around 2.5% of trade), and the feedback between the firms and the stock market.

The network increasing returns to scale of Arthur (1990), the study of imitation and social learning (Gale and Rosenthal, 1996), and the work in contagion all suggest the importance of communication networks in correlating human behavior.

B. Process Models, Dynamics, and Heterogeneous Agents

A key theme in our approach is the necessity to have a fully defined process model in the study of stock market trading. Much of the misunderstanding about rational expectations dissolves when a consistent process model of trade is considered. Recently in the study of economics, there has been a growing realization of the importance of studying models with agents that have heterogeneous characteristics (see for example Grandmont, 1993), expectations, and strategies (Arthur, Holland, LeBaron, Palmer, and Tayler, 1996; De Long, Shleifer, Summers, and Waldmann, 1990). The assumption of rational expectations or non-cooperative equilibrium, by imposing prematurely consistency conditions on expectations, removes the degrees of freedom present in the actual dynamics, which could result in many different trajectories. The empirical question of how people form expectations is finessed and replaced by a convenient mathematical simplification that has to be rationalized as a "good enough approximation".

C. Learning, Habit, and Optimization

In the work of Arthur, Holland, LeBaron, Palmer and Tayler (1996), the agents are highly adaptive, living in a soup of different conjectures and rules of thumb which they test incessantly against each other. Simon, years ago, coined the term "satisficing" to indicate behavior where habit takes over from active search. In finance there are the ideas of the market niche and financial boutiques. In the study of innovation and new products, since Schumpeter, it has been observed that it is not uncommon for a previously successful industry to die out because the lesson learned yesterday that led to success is not unlearned sufficiently fast to permit adaptation. There is a trade off between the variability in the environment and the flexibility of learning. If one can find the appropriate niche, then the amount of learning needed to survive may not be too high. In the markets the long term investors, bankruptcy experts, short sellers, day traders, and bond traders all have different niches in both location and timing. Peters (1994) in his consideration of fractal market analysis suggests different timing niches as critical to the trading structure.

D. The Importance of Time and Size Scales

In the study of physics, several different physical and time scales are identified. The laws governing interactions among the extremely small and the extremely large may differ from each other and from that which lies in between. Observations and analysis of emergent phenomena where the large scale behavior can not be predicted from the details of the small scale dynamics are the source of much of the contribution that physicists might make to the study of complex systems in economics or other fields.

In the study of economic activity, there appears to be natural lower bounds to the length of time in which a trade can take place. This represents a small scale cutoff in time. An upper bound reflects the time for which an individual is concerned with his or her trading. This represents a large scale cutoff in time. The lower bound is supplied by the minimum amount of time required to decide on a trade, transmit the decision, have it executed, and verify that it took place; the upper bound is the individual's length of life.

Much of microeconomics (and our examples studied in Section IV) is concerned with open systems. Convenient assumptions such as unlimited availability of credit and no bankruptcy are made in order to avoid having to consider conservation laws on money and credit or the boundary conditions caused by bankruptcy. Clearly, money is quantized at the smallest scales, and the total supply sets a large scale cutoff. When the economy is considered as a whole these details cannot be ignored. In economics, the scope of the model may require new considerations. Feedbacks and restorative forces working at one scale may have little, if any influence, at a different scale.

E. Quantitative and Qualitative Aspects of Uncertainty

In the models we investigate here, all exogenous uncertainty is, in essence, the same. But in the actual economy, different specialists may be regarded as different sensors and thus the exogenous events which trigger market reactions must be treated differently. For example, the assassination of a president probably impinges an all and few know how to asses the information. Information on a new court settlement concerning an insolvent firm is picked up immediately by the bankruptcy experts and few others. The successful testing of a new product is rarely of concern to fundamental long term investors.

Modern finance tends to treat all uncertainty as a lottery ticket. This treatment is valuable for calculating expectations after the odds have been assigned. The more basic problem is to assign the odds. The various markets do this through experts concerned with their interpretations of the different qualities of risk. As a first order approximation this can be modelled by assuming that the experts in different areas have a greater refinement of information in the area of their expertise than do others. If this were not the case, there would be no rationale for financial institutions specializing in different forms of risk.

VI. CONCLUDING REMARKS

Recently there has been a growth of relatively sophisticated behavioral models in economics where the analogies to stochastic mass particle behavior in physics and biology have been drawn. No attempt is made here to push analogies beyond the observation that even in the study of equilibrium a full process model is required and that simple rules of conservation are required. In particular this approach demystifies the role of money and various forms of credit which can then be regarded as well-defined different basic particles described by the laws of transformation. When this procedure of modelling economic process is adopted, the rational expectations assumption utilized in much of micro and macroeconomics, becomes merely one among an infinite number of behavioral procedures for updating a dynamic process that, in general, may not be heading to a unique stationary state. In this model the behavioral assumptions were limited to dividend acquisition and to simplistic behavior. A further extension of the type of model presented here is to have expectations of future stock prices depend on information networking and learning, or on some historical material and possibly on the predictions of some small subset of experts.

The explicit introduction of birth and death processes for the agents provides a strong forcing function or set of exogenous events constantly applied to the system. A sand pile analogy is appropriate, suppose we have a table of sand with a constant stream of sand being poured onto it. Eventually, if a macro stationary state for the system as a whole is reached, the amount of sand falling off the edges of the table will equal that of the sand being poured onto the pile. Similarly the births and deaths of individuals and products may reach balance. But that does not tell us about the micro behavior of the individual units.

In our models here we have not stressed either the overlapping generations or the expert agent aspects of an economy with a stock market. The rational expectations approach would have us believe that if some agents can perceive arbitrages that others miss, the arbitrage will quickly go away and the brighter agents will capture all profits. A more ecological viewpoint is consistent with the existence of some experts who make a good living in their niches but are not common enough and do not live long enough to capture the whole market. It is feasible that there are virtuoso players in finance as there are in music. But there is little evidence that a Mozart or a Bach can found a dynasty of progeny with equal talent for any length of time. The classical economic models which reflect knowing everything and calculating everything, while learning nothing, appear more and more to provide a poor model of stock trading. Yet at the other extreme a little introspection by these authors leads us to doubt models which stress a high amount of learning. These may err in the other direction. Genetic algorithms may have something to say about evolution in the next 2,000 generations, but not to much to say about next year's market.

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FIGURES

FIG. 1. A-particles (white) diffusing from the left and B-particles (black) diffusing from the right. A and B annihilate when they bump into each other near the center. The annihilated particles are then fed back at opposite ends of the tube.

FIG. 2. a) Stock price variation for independent noise traders who adjust their price randomly up and down with equal probability (diffusion). N=500, $p_{max} = 500$. The time, t, is the average number of updates per agent. b) A snapshot of the selling prices p_s and buying prices p_b in the market. Note the gap around the current market price $p(t) = 247$ where the number of agents is relatively low. c) Hurst plot of the range R of price variation versus time, t. The lower curve shows the local derivative of the upper curve, indicating effective exponent H of the range, R. It converges to $H = 1/4$ for large t. The irregularities for large t are of statistical nature due to the limited data set. d) Fit to Equation (4). The upper curve is $(R^2/t^{1/2})$ and the lower curve is $(R^2/(t^{1/2}\ln(t)))$. The logarithm substantially flattens the small t correction apparent in the upper curve.

FIG. 3. As figure 2, but with an additional drift D=0.05 towards the market price.

FIG. 4. The urn model. Data shown as in Figure 2. N=200, D=0.05. The agents copy existing traders when selecting new asking prices and bidding prices. The price fluctuations are much more dramatic here, and the resulting asymptotic exponent for price variations is equal to the random walk value of $H = 1/2$ rather than $H = 1/4$. Note that the width of the distribution of prices in the market (Fig. 4b) is now self-organized, rather than limited by the boundary conditions on p.

FIG. 5. Volatility feedback in the urn model. The diffusion of prices for each agent is equal to the observed price variation over the previous 50 time steps. N=1000, D=0.2. Note the wide plateau where the exponent $H = 0.65$. d) Distribution of price variations for various time intervals dt. e) Scaling plot of price fluctuations as defined in text. All curves can be described by a single scaling variable. The scaling exponent is $H = 0.65$. f) Price fluctuations for a single value of $dt = 1600$ compared with a Gaussian. Note the fat tail, indicating a significant probability of having variations exceeding several standard deviations.

FIG. 6. Urn model with 2% of the agents are fundamental value buyers, and N=500. The fundamental value traders ask and bid prices in the range between 1931 and 2068. a) Note the "bubble" around t=200000 where the market price exceeds anything that can be justified by fundamental value estimates. b) A typical configuration of prices in the market.

FIG. 7. a) Price variations for 20% fundamental value traders. Market prices are confined to the region spanned by the fundamental value traders, except for short glitches. b) Distribution of prices at the anomalous low price around t=400000. At that point, the distribution is skew, with none of the rational traders owning shares.